Lemma (Bessel's Inequality)

$(X, (\cdot,\cdot))$ inner product space, $\{x_n\} \subset X$ o.n.s. Then, $\forall x \in X$ the series $\sum_{n=1}^{\infty} |(x, e_n)|^2$ converges and

(BI) $$\underbrace{\sum_{n=1}^{\infty} |(x, e_n)|^2}_{= \|(x, e_n)\|^2_{\ell^2}} \leq \|x\|^2$$ Bessel's Inequality.

Proof. $x \in X$, $\forall k \in \mathbb{N}_+$ $y_k = \sum_{n=1}^{k} (x, e_n) e_n$

$$\|x - y_k\|^2 = (x - y_k, x - y_k) = \|x\|^2 + \|y_k\|^2 - (y_k, x) + - (x, y_k) = \|x\|^2 + \|\sum_{n=1}^{k} (x, e_n) e_n\|^2 + - \sum_{n=1}^{k} (x, e_n) \underbrace{(e_n, x)}_{= \overline{(x, e_n)}} - \sum_{n=1}^{k} \overline{(x, e_n)} (x, e_n)$$

$\sum_{n=1}^{k} (x, e_n) e_n \in Sp\{e_1, \dots, e_k\}$

By Pythagoras' Theorem $\|\sum_{n=1}^{k} (x, e_n) e_n\|^2 = \sum_{n=1}^{k} |(x, e_n)|^2$

So,

$$\|x - y_k\|^2 = \|x\|^2 + \cancel{\sum_{n=1}^{k} |(x, e_n)|^2} - \cancel{\sum_{n=1}^{k} |(x, e_n)|^2} - \sum_{n=1}^{k} |(x, e_n)|^2$$

$$\sum_{n=1}^{k} |(x, e_n)|^2 = \|x\|^2 - \underbrace{\|x - y_k\|^2}_{\geq 0} \leq \|x\|^2$$

The inequality above holds true $\forall k \in \mathbb{N}_+$.

$S_k := \sum_{n=1}^{k} |(x, e_n)|^2$ $\qquad S_k \leq S_{k+1}$, $\forall k$

$S_k \geq 0$ so we have $(S_k)_{k \in \mathbb{N}_+}$ an encreasing sequence that is bounded from above by $\|x\|^2$ $\Rightarrow$ $(S_k)_k$ is convergent to $\sup_{k \in \mathbb{N}_+} S_k$ $\Rightarrow$ $\sum_{n=1}^{\infty} |(x, e_n)|^2 = \sup_{k \in \mathbb{N}_+} S_k \leq \|x\|^2$

hence we have proved (BI).

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Theorem. H Hilbert space and $\{e_n\} \subset H$ o.n.s.
Consider $(\alpha_n)_n \in \mathbb{F}$ sequence of scalars. Then the series $\sum_{n=1}^{\infty} \alpha_n e_n$ converges $\Longleftrightarrow$ $(\alpha_n)_n \in \ell^2$ $\left(\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty \right)$.

Proof. "$\Rightarrow$" Assume that $\sum_n \alpha_n e_n$ converges to $x \in H$, that is $x = \sum_n \alpha_n e_n$. For any e_m,

$$(x, e_m) = \left(\sum_n \alpha_n e_n, e_m \right) \overset{\text{cont. }(\cdot,\cdot)}{=} \sum_n \alpha_n \underbrace{(e_n, e_m)}_{=\delta_{nm}} = \alpha_m$$

By Bessel's Inequality:

$$\sum_{m=1}^{\infty} |\alpha_m|^2 = \sum_{m=1}^{\infty} |(x, e_m)|^2 \overset{(BI)}{\leq} \|x\|^2$$

$$\Rightarrow (\alpha_n)_n \in \ell^2.$$

"$\Leftarrow$" Assume $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$, let us show that $\sum_{n=1}^{\infty} \alpha_n e_n = x \in H$.

$S_k = \sum_{n=1}^{k} \alpha_n e_n$. We will show that $\{S_k\} \subset H$ is a Cauchy sequence: for $h < k$

$$\|S_k - S_h\|^2 = \| \underbrace{\sum_{n=h+1}^{k} \alpha_n e_n}_{\in \operatorname{Sp}\{e_{h+1}, \dots, e_k\}} \|^2 \overset{\text{Pythagoras' Theorem}}{=} \sum_{n=h+1}^{k} |\alpha_n|^2 \quad (S)$$

$\tilde{S}_k = \sum_{n=1}^{k} |\alpha_n|^2$, by assumption, the seq. of partial sums $(\tilde{S}_k)_k$ is convergent to $\sum_{n=1}^{\infty} |\alpha_n|^2$
$\Rightarrow$ $(\tilde{S}_k)$ is a Cauchy sequence and by (S) we have

$$\|S_k - S_h\|^2 = \tilde{S}_k - \tilde{S}_h$$

So $\forall \varepsilon>0 \quad \exists N_0 \in \mathbb{N} \,/\, \forall k>h \geq N_0$

$\tilde{S}_k - \tilde{S}_n < \varepsilon^2 \implies \| S_k - S_n \| < \varepsilon$

Hence the sequence $\{S_k\}$ is a Cauchy seq.

since H is complete, $S_k \to x \in H$. ☒

Corollary. H Hilbert space $\{e_n\} \subset H$ o.n.s. Then $\sum_n (x,e_n)e_n$ converges, $\forall x \in H$.

Proof. By Bessel's Ineq. $\sum_n |(x,e_n)|^2 < \infty$ that is, $((x,e_n))_n \in \ell^2$ so by the previous theorem, the series $\sum_n (x,e_n)e_n$ is convergent.

Does the series $\sum_n (x,e_n)e_n$ converge to x?

The answer is NO, in general.

Example In $H=\mathbb{R}^3$ with the standard o.n.b. $\{e_1,e_2,e_3\}$. $\forall x \in \mathbb{R}^3 \quad x=(x,e_1)e_1+(x,e_2)e_2+(x,e_3)e_3$

Now consider the o.n.s $\{e_1,e_2\}$. For any $v \in \mathbb{R}^3 /$ $(v,e_3)\neq 0$, then $(v,e_1)e_1+(v,e_2)e_2 \neq v$

Definition. H Hilbert space. A sequence $\{x_n\} \subset H$ is said to be complete if $\overline{\operatorname{Sp}\{x_n\}} = H$.

Definition. H Hilbert space, $\{e_n\}$ o.n.s. of H. Then we say that $\{e_n\}$ is an orthonormal basis (o.n.b.) if $\{e_n\}$ is complete $(\Leftrightarrow \overline{\operatorname{Sp}\{e_n\}}=H)$.

Theorem (Characterization for o.n.b.)

H Hilbert space and $\{e_n\} \subset H$ o.n.s. in H.

Then the following conditions are equivalent:

1) $\{e_n\}^\perp = \{0\}$

2) $\overline{\mathrm{Sp}\{e_n\}} = H$

3) $\|x\|^2 = \sum_{n=1}^{\infty} |(x,e_n)|^2, \quad \forall x \in H$ Parseval's Theorem

4) $x = \sum_{n=1}^{\infty} (x,e_n) e_n, \quad \forall x \in H.$

this is conv. by the previous corollary

Proof. 1) => 4) $y = x - \sum_{n=1}^{\infty}(x,e_n)e_n$

$\forall m \in \mathbb{N}_+, \quad (y, e_m) = (x, e_m) - \sum_{n=1}^{\infty} (x,e_n)\underbrace{(e_n, e_m)}_{=\delta_{nm}}$

$= (x, e_m) - (x, e_m) = 0$

$\Rightarrow y \in \{e_n\}^\perp = \{0\} \Rightarrow y = 0$ and

$x = \sum_{n=1}^{\infty} (x,e_n)e_n$

4) => 2) We know $\mathrm{Sp}\{e_n\} \subseteq H \Rightarrow \overline{\mathrm{Sp}\{e_n\}} \subseteq H$

we have to prove: $H \subseteq \overline{\mathrm{Sp}\{e_n\}}$

$\forall x \in H, \quad x = \sum_{n=1}^{\infty}(x,e_n)e_n = \lim_{k\to\infty} \underbrace{\sum_{n=1}^{k}(x,e_n)e_n}_{\in \mathrm{Sp}\{e_1,\dots,e_k\} \subseteq \mathrm{Sp}\{e_n\}}$

Since $\sum_{n=1}^{k}(x,e_n)e_n \in \mathrm{Sp}\{e_n\}, \quad \forall k \in \mathbb{N}_+$

$\Rightarrow \lim_{k\to\infty} \sum_{n=1}^{k}(x,e_n)e_n \in \overline{\mathrm{Sp}\{e_n\}}$

Hence $\overline{\mathrm{Sp}\{e_n\}} = H$ ($\Leftrightarrow \{e_n\}$ is complete).

4) => 3) $\|x\|^2 = \|\sum_{n=1}^{\infty}(x,e_n)e_n\|^2$

$= \|\lim_{k\to\infty} \sum_{n=1}^{k}(x,e_n)e_n\|^2$

$\overset{\text{cont. } \|\cdot\|}{=} \lim_{k\to\infty} \|\underbrace{\sum_{n=1}^{k}(x,e_n)e_n}_{\in \mathrm{Sp}\{e_1,\dots,e_k\}}\|^2$

Pythagoras' Theorem

$$= \lim_{K\to\infty} \sum_{m=1}^{K} |(x,e_m)|^2$$

$$= \sum_{m=1}^{\infty} |(x,e_m)|^2$$

hence $\|x\|^2 = \sum_{m=1}^{\infty} |(x,e_m)|^2$.

2) $\Rightarrow$ 1) Recall: $A^\perp = \bar{A}^\perp$ (I), $A^\perp = \mathrm{Sp}\, A^\perp$ (II)

$\{e_m\}^\perp \overset{(II)}{=} \mathrm{Sp}\{e_m\}^\perp \overset{(I)}{=} \overline{\mathrm{Sp}\{e_m\}}^\perp = H^\perp = \{0\}$.

3) $\Rightarrow$ 1) $y \in \{e_m\}^\perp \Rightarrow (y,e_m)=0,\ \forall m$

by Parseval's Theorem $\|y\|^2 = \sum_m |\underbrace{(y,e_m)}_{=0}|^2 = 0$

$\Rightarrow y = 0$ $\{e_m\}^\perp = \{0\}$. ∎

Remark. H Hilbert space, $\{e_m\}$ o.n.s. in H.
Then $\{e_m\}$ is an o.n.b. $\iff$ one of the four conditions above is satisfied.

Example. $H = \ell^2$ $\{\delta_m\}_{m\in\mathbb{N}_+}$ $\delta_m = (0,0,\dots,0,\underset{m^{th}\text{ entry}}{\overset{\uparrow}{1}},0,\dots,0\dots)$

$\{\delta_m\}$ is an o.n.s. $\{\delta_m\}$ is an o.n.b.

$\forall x\in\ell^2$ $x = (x_m)$ $\Rightarrow \|x\|^2 = \sum_{m=1}^{\infty} |x_m|^2 < \infty$

$(x,\delta_m) = x_m$ $\Rightarrow$ $\sum_{m=1}^{\infty} |(x,\delta_m)|^2 = \|x\|^2$

By Parseval's Theorem $\{\delta_m\}$ is an o.n.b.

Do all Hilbert space have o.n.b. ?

The answer is NO ... there are Hilbert spaces that are "too large" to be spanned by a o.n.s.

Theorem. Finite-dimensional vector spaces are separable.

Proof (hint). Use the "standard trick":

Consider a basis $\{e_1, \dots, e_n\}$, then if the space is real consider the set $S = \left\{ \sum_{j=1}^{n} \alpha_j e_j, \; \alpha_j \in \mathbb{Q} \right\}$, S is countable and dense in the space.

If the space is complex, consider $S = \left\{ \sum_{j=1}^{n} \alpha_j e_j, \right.$ with α_j "complex rational", that is $\left. \alpha_j = q_j + i r_j, \; q_j, r_j \in \mathbb{Q} \right\}$

S is countable and dense.

Theorem. An infinite-dimensional Hilbert space H is separable $\iff$ H has an o.n.b. $\{e_n\} \subset H$.

Proof (hint). "$\Rightarrow$" Assume H separable $\Rightarrow$ $\{x_n\} \subset H$ which is dense in H. Let us construct a linearly independent sequence $\{y_n\}$ from $\{x_n\}$ by eliminating the "x_n" whic is linear combination of the previous terms. For example:

$\{x_n\} = \{x_1, x_2, \alpha x_1 + \beta x_2, x_4, \dots\}$ then

$\{y_n\} = \{x_1, x_2, x_4, \dots\}$

So $\{y_n\}$ is a lin. indep. seq. Then applying inductively the Gram-Schmidt algorithm to $\{y_n\}$ we obtain an o.n.s. $\{e_n\}$. By construction,

$$Sp\{x_n\} = Sp\{y_n\} \overset{G-S \text{ Algorithm}}{=} Sp\{e_n\}$$

hence $H = \overline{Sp\{x_n\}} = \overline{Sp\{e_n\}}$.

"$\Leftarrow$" $S_K = \left\{ \sum_{n=1}^{K} \alpha_n e_n, \right.$ with α_n rational or complex rational $\left. \right\}$

then S_K is countable and

$\bigcup_{k \in \mathbb{N}_+} S_k$ is countable (countably union of countable sets) and dense in H. ∎

Example. ℓ^2 is separable because it has the o.n.b. $\{\delta_n\}$

Exercise 5, HMW2.

Exercise. H Hilbert space and $\{e_n\} \subset H$ o.n.b. Then $\forall x \in H$ the series $\sum_n (x, e_{\sigma(n)}) e_{\sigma(n)}$ converges to x, with σ any permutation ($\sigma : \mathbb{N} \to \mathbb{N}$ bijection).

Solution. By Parseval's Theorem, $\forall x \in H$

$$\|x\|^2 = \sum_n |(x, e_n)|^2 = \sum_n |(x, e_{\sigma(n)})|^2 \quad \forall \sigma \text{ perm.}$$

absolutely conv. series $\Rightarrow$ unconditionally conv.

Exercise. Parseval's relation. H Hilbert space, $\{e_n\} \subset H$ o.n.b. Then, $\forall x, y \in H$

$$(x, y) = \sum_{n=1}^{\infty} (x, e_n)(e_n, y) \qquad \text{"Parseval's rel."}$$

Sol.
$$(x, y) = \left(\sum_{n=1}^{\infty} (x, e_n) e_n, \sum_{m=1}^{\infty} (y, e_m) e_m \right)$$
$$= \sum_{n=1}^{\infty} (x, e_n) \sum_{m=1}^{\infty} \overline{(y, e_m)} \underbrace{(e_n, e_m)}_{= \delta_{nm}}$$
$$= \sum_{n=1}^{\infty} (x, e_n) \underbrace{\overline{(y, e_n)}}_{= (e_n, y)} .$$