Lemma. X, Y normed spaces, $T \in B(X,Y)$. Then $\operatorname{Ker} T$ is closed.

Proof. $\operatorname{Ker} T = T^{-1}(\{0\})$

$\{0\}$ is closed and T is continuous $\Rightarrow$ $\operatorname{Ker} T$ is closed. ◼

Definition. $T \in L(X,Y)$ the **graph** of T is defined as
$$\mathcal{G}(T) = \{(x, Tx) : x \in X\} \subset X \times Y$$

Check as exercise: $\mathcal{G}(T)$ is a subspace of $X \times Y$

Lemma. X, Y normed spaces, $T \in B(X,Y)$ then $\mathcal{G}(T)$ is closed.

Proof. Consider $(x_n, Tx_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{G}(T)$ such that $(x_n, Tx_n) \to (x,y) \in X \times Y$

We want to prove that $(x,y) \in \mathcal{G}(T) \Leftrightarrow y = Tx$

Since $x_n \to x$ in X $\Rightarrow$ $Tx_n \to Tx$ in Y because T is continuous and
$$\|(x_n, Tx_n) - (x, Tx)\|_{X\times Y} = \|x_n - x\|_X + \|Tx_n - Tx\|_Y \to 0$$
as $n \to \infty$

hence, by the uniqueness of the limit, $(x,y) = (x, Tx)$ that is $Tx = y$. ◼

Theorem (Density Principle)

X normed space, $W \subset X$ subspace and W is dense in X. Y Banach space and $S \in B(W,Y)$. Then there exists and is unique $T \in B(X,Y)$:

1) $Tw = Sw, \quad \forall w \in W$

2) $\|T\|_{B(X,Y)} = \|S\|_{B(W,Y)}$.

T is called the continuous extension of S to the bigger space X.

Proof. Given $S \in B(W,Y)$ we define the extension of S as follows: $\forall x \in X$ then by the density of W in X there exists a sequence $\{w_n\} \subset W$ such that $w_n \to x$.

$$Tx := \lim_{n \to \infty} Sw_n .$$

We have to show:

1) the limit exists in Y

2) " " does not depend on the sequence $\{w_n\}$

3) T is linear

4) T is bounded

5) T extends S ($Tw = Sw, \ \forall w \in W$ and $\|T\| = \|S\|$)

6) T is unique.

1) $\{w_n\} \subset S$: $w_n \to x$ is a Cauchy seq.

$\{Sw_n\} \subset Y$ is a Cauchy seq. in Y:

$$\|Sw_m - Sw_n\|_Y = \|S(w_m - w_n)\|_Y \le \|S\|_{B(W,Y)} \|w_m - w_n\|_X < \varepsilon$$

provided that $n, m \ge N_0 \in \mathbb{N}_+$ big enough since $\{w_n\}$ is a Cauchy seq.

Since $\{Sw_n\} \subset Y$ is a Cauchy seq. and Y is complete $\Rightarrow$ $Sw_n \to y \in Y$. We define $Tx = y$.

2) $\{w'_n\} \subset W$: $w'_n \xrightarrow[\text{lim.}]{} x$ in X, then

$$\| Sw'_n - Sw_n \|_Y = \| S(w'_n - w_n) \|_Y \underset{\text{bound.}}{\leq} \|S\| \, \|w'_n - w_n\|_X$$

$$\to 0, \quad n \to \infty$$

$$\| w'_n - w_n \|_X = \| w'_n - x + x - w_n \|_X \leq \|w'_n - x\|_X + \|x - w_n\|_X$$

$$\to 0, \quad n \to \infty$$

since $w'_n \to x$ and $w_n \to x$.

Hence $\lim_{n\to\infty} Sw'_n = \lim_{n \to \infty} Sw_n = Tx$

3) $\forall \alpha, \beta \in \mathbb{F}$, $\forall x_1, x_2 \in X$ $\quad w^1_n \to x_1$, $w^2_n \to x_2$

$\{w^1_n\}, \{w^2_n\} \subset W$ and $\{\alpha w^1_n + \beta w^2_n\} \subset W$

with $\alpha x_1 + \beta x_2 = \lim_{n\to\infty} (\alpha w^1_n + \beta w^2_n)$

$$T(\alpha x_1 + \beta x_2) = \lim_{n \to \infty} S(\alpha w^1_n + \beta w^2_n)$$

$$\overset{\text{lin}}{=} \alpha \lim_{n\to\infty} Sw^1_n + \beta \lim_{n\to\infty} Sw^2_n$$

$$= \alpha T x_1 + \beta T x_2$$

4) $\forall x \in X$, $\exists \{w_n\} \subset W$: $w_n \to x$

$$\| Sw_n \|_Y \underset{\text{bound}}{\leq} \|S\|_{B(W,Y)} \|w_n\| \qquad (*)$$

$Tx = \lim_{n\to\infty} Sw_n$ and $\|Tx\|_Y = \lim_{n\to\infty} \|Sw_n\|$

recall $\|w_n\|_X \to \|x\|$

then taking the limit for $n \to \infty$ in $(*)$

$$\|Tx\|_Y \leq \|S\|_{B(W,Y)} \|x\|, \quad \forall x \in X$$

In particular, $\|T\|_{B(X,Y)} \leq \|S\|_{B(W,Y)}$.

5) T extends S:

$\forall w \in W$ consider $w_n = w$, $\forall n \in \mathbb{N}$

then $w_n \to w$ and

$$Tw = \lim_{n\to\infty} S w_n = \lim_{n\to\infty} Sw = Sw, \quad \forall w \in W$$

We know by item 4) that $\|T\|_{B(X,Y)} \leq \|S\|_{B(W,Y)}$

let us show $\|S\|_{B(W,Y)} \leq \|T\|_{B(X,Y)}$

$\forall w \in W$ $\quad w_n = w \to w \quad Tw_n \to Tw = Sw$

bound. for T

$$\|Sw\|_Y = \|Tw\|_Y \leq \|T\|_{B(X,Y)} \|w\|_X, \quad \forall w \in W$$

hence $\|S\|_{B(W,Y)} \leq \|T\|_{B(X,Y)}$.

6) The uniqueness of T follows immediately by the uniqueness of the limit:

$$\forall x \in X, \quad Tx = \lim_{n\to\infty} S w_n, \quad \{w_n\} \subset S : w_n \to x$$

In fact, assuming $\exists\, T_2 \in B(X,Y)$ which extends S

$T_2 w = Sw$, $\forall w \in W$, then $\forall x \in X$,

$x = \lim_{n\to\infty} w_n$, $\{w_n\} \subset W$, since T_2 is

bounded $\quad T_2 x = \lim_{n\to\infty} T_2 w_n = \lim_{n\to\infty} S w_n = Tx$

hence $T_2 x = Tx$, $\forall x \in X$. $\blacksquare$

Not every linear operator is bounded.

Example. $C^1_{\mathbb{R}}([0,1]) = \{ f : [0,1] \to \mathbb{R} : f$ is differentiable and f' is continuous on $[0,1]\}$

Hence $C^1_{\mathbb{R}}([0,1]) \subset C_{\mathbb{R}}([0,1])$

$C^1_{\mathbb{R}}([0,1])$ is a subspace of $C_{\mathbb{R}}([0,1])$

$$(\alpha f+\beta g)' = \alpha f' + \beta g'$$

so $\alpha f + \beta g \in C^1_{\mathbb{R}}([0,1])$ if $f, g \in C^1_{\mathbb{R}}([0,1])$

hence $(C^1_{\mathbb{R}}([0,1]), \| \ \|_\infty)$ is a normed space.

Consider $Tf = f'(0)$, $\forall f \in C^1_{\mathbb{R}}([0,1])$

Then $T : C^1_{\mathbb{R}}([0,1]) \to \mathbb{R}$ well defined

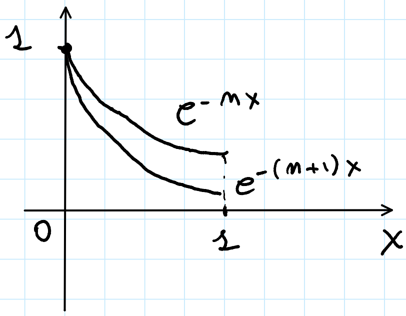
T is linear $T(\alpha f+\beta g) = (\alpha f+\beta g)'(0) = \alpha f'(0) + \beta g'(0)$

$$= \alpha T f + \beta T g$$

T is not bounded. By contradiction, assume that T is bounded: $\exists C > 0$:

(B) $\quad |Tf| \leq C \|f\|_\infty$, $\quad \forall f \in C^1_{\mathbb{R}}([0,1])$

Consider the sequence $f_n(x) = e^{-nx}$, $n \in \mathbb{N}_+$



$f_n \in C^1_{\mathbb{R}}([0,1])$, $\forall n$

$f_n'(x) = -n e^{-nx}$

$f_n'(0) = -n$

the boundedness estimate in (B) becomes:

$$|f_n'(0)| \leq C \|f_n\|_\infty, \quad \forall n$$

$\|f_n\|_\infty = \max_{x \in [0,1]} e^{-nx} = 1$, $\quad \forall n$, $\quad$ hence (B):

$$n \leq C, \quad \forall n \in \mathbb{N}_+$$

this is a contradiction, hence T is unbounded.

Let us define on $C^1_{\mathbb{R}}([0,1])$ the norm:

$\|f\|_{C^1} := \|f\|_\infty + \|f'\|_\infty$

check that $\| \cdot \|_{C^1}$ is a norm on $C^1_{\mathbb{R}}([0,1])$

Then $T : \left(C^1_{\mathbb{R}}([0,1]), \|\cdot\|_{C^1}\right) \longrightarrow \mathbb{R}$

is bounded:

$$|Tf| = |f'(0)| \le \|f'\|_\infty \le \|f'\|_\infty + \underbrace{\|f\|_\infty}_{\ge 0} = \|f\|_{C^1}$$

hence $|Tf| \le \|f\|_{C^1}$, $\forall f \in C^1_{\mathbb{R}}([0,1])$.

Def. X, Y normed spaces, $T : X \to Y$ linear.
T is called an isometry if

$$\|Tx\|_Y = \|x\|_X, \quad \forall x \in X \qquad (I)$$

Remark 1) If T is an isometry then T is bounded (by the equality in (I)).

2) $\|T\|_{B(X,Y)} = 1$

Example On every normed space we have at least an isometry. For example take $Ix = x$, $\forall x \in X$
$I : X \to X$ and $\|Ix\|_X = \|x\|_X$.

Example. $X = Y = \ell^2$, $\forall x = (x_n) \in \ell^2$
Unilateral shift $Sx = (0, x_1, x_2, \dots, x_n, \dots)$
then $S : \ell^2 \to \ell^2$, S linear, S isometry.

Solution. $x = (x_n) \in \ell^2$

$$\|Sx\|^2_{\ell^2} = \|(0, x_1, x_2, \dots, x_n, \dots)\|^2_{\ell^2} = |0|^2 + \sum_{n=1}^{\infty} |x_n|^2$$
$$= \|x\|^2_{\ell^2} < \infty$$

$S : \ell^2 \to \ell^2$ and $\|Sx\|_{\ell^2} = \|x\|_{\ell^2}$, $\forall x \in \ell^2$

S is linear: $\forall \alpha, \beta \in \mathbb{F}$, $\forall x = (x_n), y = (y_n) \in \ell^2$

$$S(\alpha x + \beta y) = (0, \alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots, \alpha x_n + \beta y_n, \dots)$$

$= \alpha(0, x_1, x_2, \dots, x_n, \dots) + \beta(0, y_1, y_2, \dots, y_n, \dots)$

$= \alpha Sx + \beta Sy$

Remark. $T : X \to Y$, T isometry then T is one-to-one

$\forall\, x_1 \neq x_2$, $x_1, x_2 \in X$ $\quad \|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \overset{(I)}{=} \|x_1 - x_2\|_X \neq 0$

hence $Tx_1 \neq Tx_2$.

If T is an isometry, in general T is not onto.

Take for instance the unilateral shift S

$$\underbrace{S\ell^2 = \{(0, x_1, \dots, x_n, \dots),\ \forall x = (x_n) \in \ell^2\}}\ \subsetneq \ell^2$$

because you can always find $y = (y_n) \in \ell^2 \,/\, y_1 \neq 0$

$y_n = \frac{1}{n}$, $n \in \mathbb{N}_+$ $\quad y_1 = 1 \neq 0$ and $(y_n) \in \ell^2$

since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

Remark. If X is a finite-dimension vector space, then if $T : X \to X$ is an isometry

$\Rightarrow$ T is onto.

$$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \underbrace{\dim X}_{=n} \qquad (**)$$

since T is an isometry $\Rightarrow$ T is one-to-one

$\Rightarrow \operatorname{Ker} T = \{0\}$ $\qquad \dim \operatorname{Ker} T = 0$

So by $(**)$ we get $\dim \operatorname{Im} T = n$

$\operatorname{Im} T \subseteq X$ $\qquad \dim \operatorname{Im} T = \dim X$

$\Rightarrow \operatorname{Im} T = X$.

So T is onto.