Exercise Consider $h \in \mathcal{C}([0,1])$ and define

$$T_h f(x) = h(x) f(x), \qquad \forall f \in L^2([0,1])$$

(T_h multiplication operator by h)

- Show that T_h is well defined and $T_h \in B(L^2([0,1]))$
- Show that if $h(x) \neq 0$, $\forall x \in [0,1]$ then T_h is invertible with inverse $T_h^{-1} = T_{\frac{1}{h}}$.

Solution. $\forall f \in L^2([0,1])$,

$$\|T_h f\|_2^2 = \int_0^1 |T_h f(x)|^2 dx = \int_0^1 |h(x)|^2 \cdot |f(x)|^2 dx$$
$$\leq \|h\|_\infty^2 \int_0^1 |f(x)|^2 dx$$
$$= \|h\|_\infty^2 \; \|f\|_2^2 < \infty$$

- $T_h : L^2([0,1]) \longrightarrow L^2([0,1])$
- $\|T_h f\|_2 \leq \|h\|_\infty \|f\|_2, \qquad \forall f \in L^2([0,1])$
- T_h is linear: $\forall f_1, f_2 \in L^2([0,1])$, $\forall \alpha, \beta \in \mathbb{F}$

$$T_h(\alpha f_1 + \beta f_2)(x) = h(x)(\alpha f_1 + \beta f_2)(x) = \alpha h(x) f_1(x) +$$
$$+ \beta h(x) f_2(x) = \alpha T_h f_1(x) + \beta T_h f_2(x)$$

- Observe that if $h(x) \neq 0$, $\forall x \in [0,1]$, $\frac{1}{h(x)} \in \mathcal{C}([0,1])$

$\Rightarrow T_{\frac{1}{h}} \in B(L^2([0,1]))$

$$T_{\frac{1}{h}} T_h f(x) = T_{\frac{1}{h}}(T_h f)(x) = \frac{1}{h(x)} T_h f(x) = \frac{1}{\cancel{h(x)}} \cancel{h(x)} f(x),$$

$\forall f \in L^2(\mathbb{R}^d)$

$$T_h T_{\frac{1}{h}} f(x) = T_h(T_{\frac{1}{h}} f)(x) = h(x) T_{\frac{1}{h}} f(x) = \cancel{h(x)} \frac{1}{\cancel{h(x)}} f(x)$$

hence $T_{\frac{1}{h}} T_h = T_h T_{\frac{1}{h}} = I_{L^2([0,1])}$ so that $T_{\frac{1}{h}} = T_h^{-1}$.

Remark. 1) In general it is not easy to find whether a bounded op. is invertible,

2) If $\dim X < \infty$ and $T \in B(X)$, then T can be represented by means of a matrix A and T is invertible $\iff \det A \neq 0$.

Unfortunately, there is no obvious generalization of this to infinite dimensional spaces. So we look for other methods for the study of the invertibility.

- First approach: look at the norm of an operator.

Theorem (Neumann series)

X Banach space. If $T \in B(X)$ with $\|T\|_{B(X)} < 1$ then $I - T \in B(X)$ is invertible and its inverse is

(NS) $$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$$ Neumann series

(convention $T^0 = I$ identity op.).

Proof. Let us first show that $\sum_{n=0}^{\infty} T^n$ is convergent to $S \in B(X)$.

Observe that X is a Banach space $\Rightarrow B(X) = B(X, \overset{\text{Banach}}{X})$ is a Banach space, hence absolutely convergent series in $B(X)$ are convergent in $B(X)$.

So we study the absolute convergence of $\sum_{n=0}^{\infty} T^n$

$$\|T^n\|_{B(X)} = \|\underbrace{T \cdots T}_{n\text{-times}}\|_{B(X)} \leq \|T\|^n_{B(X)}$$

$$\sum_{n=0}^{\infty} \|T\|^n_{B(X)} < \infty \quad \text{since} \quad \|T\|_{B(X)} < 1$$

Since $\|T^n\|_{B(X)} \leq \|T\|^n_{B(X)}$, by the comparison theorem for non-negative series of scalars, $\sum_{n=0}^{\infty} \|T^n\|_{B(X)} < \infty$ hence $\sum_{n=0}^{\infty} T^n$ is convergent to $S \in B(X)$, that is $S = \sum_{n=0}^{\infty} T^n$.

Secondly, we show $(I-T)S = I$.

Define $S_K = \sum_{n=0}^{K} T^n$, let us evaluate

$$\|(I-T)S_K - I\|_{B(X)} = \|(I-T)(I+T+T^2+\dots+T^K) - I\|_{B(X)}$$
$$= \|-T^{K+1}\|_{B(X)} = \|T^{K+1}\|_{B(X)} \leq \|T\|^{K+1}_{B(X)}$$

$\lim_{K\to\infty} \|T\|^{K+1}_{B(X)} = 0$, by the comparison theorem

$$\lim_{K\to\infty} \|(I-T)S_K - I\|_{B(X)} = 0$$

$\|$ (continuity of norm)

$$\|(I-T)\underbrace{\lim_{K\to\infty} S_K}_{=S} - I\|_{B(X)} = \|(I-T)S - I\|_{B(X)}$$

if $\|(I-T)S - I\|_{B(X)} = 0 \iff (I-T)S = I$

Similarly $S(I-T) = I$. ∎

Example. Consider $k(x,y) = A\sin(x-y)$, $A \in \mathbb{C}$, $|A| < 1$.

Then, for any $f \in C([0,1])$ $\exists\, g \in C([0,1])$ |

$$g(x) = f(x) + \int_0^1 k(x,y)\, g(y)\, dy. \qquad (A1).$$

Solution. Define $Tf(x) = \int_0^1 k(x,y) f(y)\, dy$

Since $k \in C([0,1]\times[0,1])$ we know that

$T \in B(C([0,1]))$ with $\|Tf\|_\infty \leq \|k\|_\infty (1-0)\|f\|_\infty$

that is, $\|Tf\|_\infty \leq \|k\|_\infty \|f\|_\infty$, $\forall f \in C([0,1])$

hence $\|T\|_{B(X)} \leq \|k\|_\infty$

$$\|K\|_\infty = \sup_{(x,y)\in[0,1]\times[0,1]} |K(x,y)| = \sup_{(x,y)\in[0,1]\times[0,1]} |A\sin(x-y)| \le |A|$$

hence $\|T\|_{B(X)} \le |A| < 1$ by assumption.

rewrite (A1) as :

$$Ig = f + Tg \qquad (A1)$$

$(I-T)g = f$ so by the Neumann series ($X = \mathcal{C}([0,1])$ with $\|\cdot\|_\infty$ is a Banach space) $(I-T)$ is invertible and

$$g = (I-T)^{-1} f = \left(\sum_{n=0}^{\infty} T^n\right) f \in \mathcal{C}([0,1]).$$

Remark. The theorem on Neumann series is valid also for $I + T \in B(X)$, with $\|T\|_{B(X)} < 1$

with $(I+T)^{-1} = (I-(-T))^{-1}$

$\tilde{T} := -T$, then $\|\tilde{T}\|_{B(X)} = \|-T\|_{B(X)} = \|T\|_{B(X)} < 1$

use the Neumann series for $\tilde{T}$:

$$(I-\tilde{T})^{-1} = \sum_{n=0}^{\infty} (\tilde{T})^n$$

hence

$$(I+T)^{-1} = \sum_{n=0}^{\infty} (-T)^n = \sum_{n=0}^{\infty} (-1)^n T^n.$$

Second method for the study of the invertibility:

If T is invertible then T is a bijection and $T^{-1} \in B(Y,X)$ if $T \in B(X,Y)$

Recall that if T is a bijection, this does not imply in general that T^{-1} is bounded.

Theorem (Open mapping theorem).
X, Y Banach spaces, $T \in B(X,Y)$ is onto.
Then T is an open mapping (that is, if $A \subseteq X$ open set $\Rightarrow$ $T(A) \subseteq Y$ is an open set in Y).

Corollary (Banach Isomorphism Theorem)
X, Y Banach spaces, $T \in B(X,Y)$ is a bijection.
Then T is invertible ($T^{-1} \in B(Y,X)$).

Proof. By the Open mapping theorem T is an open mapping. We have to show: T^{-1} is continuous
$\Leftrightarrow$ $\forall A \subseteq X$ open set in X $\Rightarrow$ $\underbrace{(T^{-1})^{-1}}_{=T}(A)$ is an open set in Y
this is true since T is an open mapping!

Corollary (Closed Graph Theorem).
X, Y Banach spaces and $T \in L(X,Y)$ such that $\mathcal{G}(T) = \{(x, Tx), x \in X\} \subset X \times Y$ is closed. Then $T \in B(X,Y)$.

Proof. Since X and Y are Banach spaces, $X \times Y$ is a Banach space with the norm
$\|(x,y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$ (check as exercise)
$\mathcal{G}(T) \subset X \times Y$ is a closed subspace $\Rightarrow$ $\mathcal{G}(T)$ is a Banach space with the induced norm.
We define $R : \mathcal{G}(T) \to X$ by
$$R(x, Tx) = x$$
R is one-to-one and onto, moreover

$$\|R(x,Tx)\|_X = \|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|(x,Tx)\|_{\mathcal{G}(T)}$$

hence R is bounded and by Banach Isomorphism Theorem R is invertible, $R^{-1}: X \to \mathcal{G}(T)$

$\exists\, C>0$: $\|R^{-1}x\|_{\mathcal{G}(T)} \leq C\|x\|_X$

$R^{-1}x = (x, Tx)$ hence $\|(x,Tx)\|_{\mathcal{G}(T)} \leq C\|x\|_X$

Now: $\|Tx\|_Y \leq \|Tx\|_Y + \|x\|_X = \|(x,Tx)\|_{\mathcal{G}(T)} \leq C\|x\|_X$

hence $\|Tx\|_Y \leq C\|x\|_X$, $\forall x \in X$

that is, T is bounded.

Exercise. Assume X vector space with $\|\cdot\|_1$, $\|\cdot\|_2$ two norms on X such that $(X, \|\cdot\|_i)$ is a Banach space for $i=1,2$. Assume $\exists\, C>0$:

$$\|x\|_1 \leq C\|x\|_2, \quad \forall x \in X \quad \odot .$$

Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms.

Solution. Consider $I: (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$

$Ix = x$, $I \in \mathcal{B}((X,\|\cdot\|_2), (X,\|\cdot\|_1))$

by $\odot$. I is a bijection $\Rightarrow$ By the Banach isomorphis theorem $I^{-1} = I \in \mathcal{B}((X,\|\cdot\|_1),(X,\|\cdot\|_2))$

$\Leftrightarrow \exists\, \tilde{C}>0$: $\|x\|_2 \leq \tilde{C}\|x\|_1$, $\forall x \in X$.

So the two norms are equivalent.

Lemma 1. If X, Y are normed spaces and $T \in \mathcal{B}(X,Y)$ is invertible then $\forall x \in X$, $\|Tx\| \geq \|T^{-1}\|^{-1}_{\mathcal{B}(Y,X)} \|x\|$.

Proof. $\|x\|_X = \|T^{-1}Tx\|_X \leq \|T^{-1}\|_{\mathcal{B}(Y,X)} \|Tx\|_Y$

hence $\|Tx\|_Y \geq \|T^{-1}\|^{-1}_{\mathcal{B}(Y,X)} \|x\|_X$.

Lemma 2. X Banach space, Y normed space and $T \in B(X,Y)$. If $\exists\, \alpha > 0$ such that $\|Tx\|_Y \geq \alpha \|x\|_X$, $\forall x \in X$, then $\operatorname{Im} T$ is closed.

Proof. Take a sequence $\{y_n\} \subset \operatorname{Im} T$ such that $y_n \to y \in Y$ and show $y \in \operatorname{Im} T$.

Since $\{y_n\} \subset \operatorname{Im} T$ $\exists\, \{x_n\} \subset X$ such that $Tx_n = y_n$. $\{y_n\}$ is a Cauchy seq. since it is convergent. $\{x_n\}$ is a Cauchy sequence

$$\underset{\displaystyle \|y_n - y_m\|_Y < \varepsilon}{\underset{\shortparallel}{\|Tx_n - Tx_m\|_Y}} = \|T(x_n - x_m)\|_Y \geq \alpha \|x_n - x_m\|_X$$

provided that $n, m \geq N_0$

Hence $\{x_n\} \subset X$ is a Cauchy seq. $\Rightarrow$ $x_n \to x \in X$ because X is complete.

Since T is continuous $Tx_n \to Tx$ so $y = Tx$ by the uniqueness of the limit.

Characterization of invertible operators

Given X, Y Banach spaces and $T \in B(X,Y)$ then the following conditions are equivalent:

1) T is invertible

2) $\operatorname{Im} T$ is dense Y and $\exists\, \alpha > 0 : \|Tx\|_Y \geq \alpha \|x\|_X$, $\forall x \in X$.

Proof. 1) $\Rightarrow$ 2) T is invertible hence T is a bijection so $\operatorname{Im} T = Y$ and by Lemma 1 with $\alpha = \|T^{-1}\|^{-1}$ we have $\|Tx\|_Y \geq \alpha \|x\|_X$.

2) $\Rightarrow$ 1) By Lemma 2, $\operatorname{Im} T$ is closed: $\overline{\operatorname{Im} T} = \operatorname{Im} T$. By assumption $\operatorname{Im} T$ is dense in $Y \Rightarrow Y = \overline{\operatorname{Im} T} = \operatorname{Im} T$ hence T is onto. By assumption, $\|Tx\|_Y \geq \alpha \|x\|_X, \ \forall x$. So if $Tx = 0$

$\Leftrightarrow \|Tx\|_Y = 0 \Rightarrow \alpha \|x\|_X = 0 \Leftrightarrow \|x\|_X = 0$

$\Leftrightarrow x = 0$ hence T is one-to-one.

Hence T is invertible by the Banach isomorphism theorem.