Lecture 17

Characterization for invertible operators

X, Y Banach spaces, $T \in B(X,Y)$, then t.f.c.a.e.

1) T is invertible
2) $\operatorname{Im} T$ is dense in Y and $\exists \alpha > 0 : \|Tx\|_Y \geq \alpha \|x\|_X, \forall x$

Corollary.

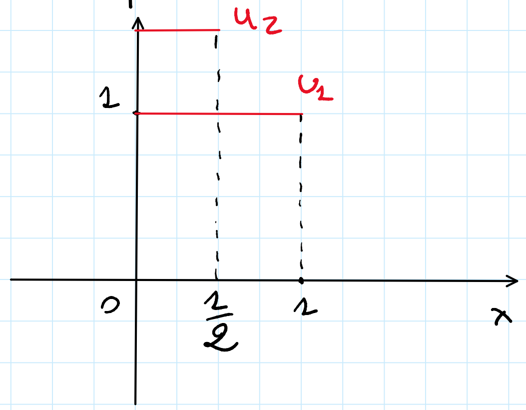
X, Y Banach spaces, $T \in B(X,Y)$, then t.f.c.a.e.:

1) T is not invertible
2) $\operatorname{Im} T$ is not dense in Y or $\exists \{x_n\} \subset X : \|x_n\|_X = 1$ and $Tx_n \to 0$.

Example. Consider $T_h \in B(L^2([0,1]))$ defined by $T_h f(x) = h(x) f(x)$, $\forall f \in L^2([0,1])$, whereas $h \in C([0,1])$. Consider $h(x) = x$, $\forall x \in [0,1]$ then T_h is not invertible.

Recall: if $h(x) \neq 0$, $\forall x \in [0,1]$ then T_h is invertible and $T_h^{-1} = T_{\frac{1}{h}}$.

Solution. We show that T_h is not invertible by means of condition 2) of the previous corollary. Consider the sequence $u_n(x) = \sqrt{n}\, \chi_{[0,\frac{1}{n}]}$, $n \in \mathbb{N}_+$



$$\|u_n\|_2^2 = \int_0^1 n\,\chi_{[0,\frac{1}{n}]}(x)\,dx = n\int_0^{\frac{1}{n}} dx = n\cdot\frac{1}{n} = 1, \quad \forall n$$

$$T_h u_n(x) = x\,u_n(x) = x\sqrt{n}\,\chi_{[0,\frac{1}{n}]}$$

$$\|T_h u_n\|_2^2 = \int_0^1 x^2 n\,\chi_{[0,\frac{1}{n}]}(x)\,dx = n\int_0^{\frac{1}{n}} x^2\,dx$$

$$= n\left[\frac{x^3}{3}\right]_0^{\frac{1}{n}} = \frac{\cancel{n}}{3}\,\frac{1}{n^{\cancel{3}\,2}} = \frac{1}{3n^2}$$

$$\|T_h u_n\|_2 = \frac{1}{\sqrt{3}\,n} \longrightarrow 0 \quad \text{as} \quad n\to\infty$$

So by the previous corollary T_h is not invertible.

Example. Consider the operator $T:\ell^2\to\ell^2$ defined by

$$Tx = \left((1+1)x_1, \left(1+\frac{1}{2}\right)x_2, \dots, \left(1+\frac{1}{n}\right)x_n, \dots\right),$$

$$\forall x = (x_n)\in\ell^2.$$

Show that $T\in B(\ell^2)$ and T is invertible.

Solution. $Tx = T_c x$, with $c = \left(1+\frac{1}{n}\right)_{n\in\mathbb{N}_+}$

$c\in\ell^\infty$ and $\|c\|_{\ell^\infty} = \sup\limits_{n\in\mathbb{N}_+}\left(1+\frac{1}{n}\right) = 2$

$$\|Tx\|_{\ell^2}^2 = \sum_{n=1}^{\infty}\left|\left(1+\frac{1}{n}\right)x_n\right|^2 \le 4\sum_{n=1}^{\infty}|x_n|^2 = 4\|x\|_{\ell^2}^2$$

$$\|Tx\|_{\ell^2} \le 2\|x\|_{\ell^2}, \quad \forall x\in\ell^2$$

T is linear (check) $\Rightarrow T\in B(\ell^2)$.

Let us show that T is a bijection

- T is one-to-one $\quad Tx = 0 \iff$

$$\underbrace{\left(1+\frac{1}{n}\right)}_{>0}x_n = 0, \quad \forall n\in\mathbb{N}_+$$

$$\iff x_n = 0, \quad \forall n\in\mathbb{N}_+$$

$$\iff x = 0$$

- T is onto. $\forall y = (y_n) \in \ell^2$ we have to find $x = (x_n) \in \ell^2$: $Tx = y$. $\iff$

$$\left(1+\frac{1}{n}\right)x_n = y_n, \quad \forall n \in \mathbb{N}_+$$

$$\iff x_n = \frac{n}{n+1} y_n, \quad \forall n \in \mathbb{N}_+$$

$$x = \left(\frac{n}{n+1} y_n\right)_{n \in \mathbb{N}_+} \overset{?}{\in} \ell^2$$

$$\|x\|_{\ell^2}^2 = \sum_{n=1}^{\infty} \underbrace{\frac{n^2}{(n+1)^2}}_{\leq 1} |y_n|^2 \leq \sum_{n=1}^{\infty} |y_n|^2 = \|y\|_{\ell^2}^2 < \infty$$

So T is onto $\Rightarrow$ T is a bijection and by the Banach isomorphism theorem we get T is invertible.

Example. $T : \ell^2 \to \ell^2$ given by, $\forall x = (x_n) \in \ell^2$

$$Tx = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, \dots\right)$$

- Show $T \in \mathcal{B}(\ell^2)$
- Is T invertible?

Solution. $c = \left(\frac{1}{n}\right)_{n \in \mathbb{N}_+}$ $\quad c \in \ell^\infty \quad \|c\|_{\ell^\infty} = 1$

$\Rightarrow T = T_c \in \mathcal{B}(\ell^2)$ (see the previous proof.)

- T is one-to-one : $Tx = 0 \iff$

$$\frac{x_n}{n} = 0, \ \forall n \in \mathbb{N}_+ \iff$$
$$x_n = 0, \ \forall n \in \mathbb{N}_+ \iff$$
$$x = 0$$

- $\forall y = (y_n) \in \ell^2$, we study $Tx = y \iff$

$$\frac{x_n}{n} = y_n, \ \forall n \in \mathbb{N}_+ \iff$$
$$x_n = n y_n, \ \forall n \in \mathbb{N}_+$$

Observe that in general $x = (n y_n)_{n \in \mathbb{N}_+}$ does not belong to ℓ^2. Ex. $y = \left(\frac{1}{n}\right)_{n \in \mathbb{N}_+} \in \ell^2$ then

$x_n = n \cdot \frac{1}{n} = 1, \quad \forall n$

$x = (1)_n \notin \ell^2$

Hence T is not onto $\Rightarrow$ T is not invertible.

Application of the Closed Graph Theorem:

Theorem (Uniform Boundedness Principle)

X, Y Banach spaces. Suppose that A is a non-empty set and $\forall \alpha \in A$, $T_\alpha \in B(X, Y)$. If $\forall x \in X$ the set $\{ \|T_\alpha x\|_Y, \alpha \in A \}$ is bounded then the $\{ \|T_\alpha\|_{B(X,Y)}, \alpha \in A \}$ is bounded.

Corollary. X, Y Banach spaces, $\{T_n\} \subset B(X, Y)$ such that $\forall x \in X$, $\{ \|T_n x\|_Y, n \in \mathbb{N} \}$ is bounded, then $\{ \|T_n\|_{B(X,Y)}, n \in \mathbb{N} \}$ is bounded.

Corollary. X, Y Banach spaces, $\{T_n\} \subset B(X, Y)$. Suppose that $\lim_{n \to \infty} T_n x$ exists $\forall x \in X$ and define

$Tx := \lim_{n \to \infty} T_n x$. Then $T \in B(X, Y)$.

Proof. Since by assumption $\lim_{n \to \infty} T_n x$ exists, $\{T_n x\}$ is a convergent sequence hence it is bounded: $\{ \|T_n x\|_Y, n \in \mathbb{N} \}$ is bounded.

Then by the UBP we have $\{ \|T_n\|_{B(X,Y)}, n \in \mathbb{N} \}$

is bounded: $\exists M > 0$: $\|T_n\|_{B(X,Y)} \le M, \ \forall n$

$$\|Tx\|_Y = \|\lim_{n\to\infty} T_n x\|_Y \overset{\text{cont. } \|\cdot\|}{=} \lim_{n\to\infty} \|T_n x\|_Y \le$$

$$\le \lim_{n\to\infty} \|T_n\|_{B(X,Y)} \|x\|_X \le \lim_{n\to\infty} M \|x\|_X$$

$$= M\|x\|_X$$

hence $\|Tx\|_Y \le M\|x\|_X, \quad \forall x \in X$

T is linear: $\forall \alpha, \beta \in \mathbb{F}, \quad \forall x_1, x_2 \in X$,

$$T(\alpha x_1 + \beta x_2) = \lim_{n\to\infty} T_n(\alpha x_1 + \beta x_2) = \alpha \lim_{n\to\infty} T_n x_1 + \beta \lim_{n\to\infty} T_n x_2$$
$$= \alpha T x_1 + \beta T x_2 .$$

Hence $T \in B(X, Y)$.

Dual spaces

X normed space, $\quad X' = B(X, \mathbb{F})$

Theorem. If X is a finite-dimensional normed space with basis $\{v_1, \dots, v_n\}$. Then X' has a basis $\{f_1, \dots, f_n\}$ such that

(C) $\qquad f_j(v_k) = \delta_{jk}, \qquad \forall j, k = 1, \dots, n$

In particular, $\dim X' = \dim X$.

Proof. Since $\{v_1, \dots, v_n\}$ is a basis for X, $\forall x \in X$ $\exists!\ \alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $x = \sum_{k=1}^n \alpha_k v_k$.

Define, $\forall j = 1, \dots, n$, the functional f_j as follows:

$$f_j(x) = f_j\left(\sum_{k=1}^n \alpha_k v_k\right) \overset{f_j \text{ lin}}{=} \sum_{k=1}^n \alpha_k \underbrace{f_j(v_k)}_{= \delta_{jk} \text{ by (C)}} = \alpha_j$$

By construction f_j is linear, $\forall j=1,\dots,n$ since $\dim X = n < \infty$ and $f_j : X \to \mathbb{F}$ linear $\Rightarrow$ f_j is bounded that is $f_j \in X'$, $\forall j=1,\dots,n$. Let us show $\{f_1,\dots,f_n\} \subset X'$ is a basis. $\forall \alpha_1,\dots,\alpha_n \in \mathbb{F}$ such that

$$\sum_{j=1}^{n} \alpha_j f_j = 0 \qquad (0\text{-functional})$$

$\forall k=1,\dots,n$

$$\Big(\sum_{j=1}^{n} \alpha_j f_j\Big)(v_k) = 0$$

$$\| $$

$$\sum_{j=1}^{n} \alpha_j \underbrace{f_j(v_k)}_{=\delta_{jk}} = \alpha_k$$

hence $\alpha_k = 0$, $\forall k=1,\dots,n$

$\Rightarrow$ $\{f_1,\dots,f_n\}$ is a linearly independent set

From linear algebra: if $\dim X = n$, $\dim Y = m$ then $\dim L(X,Y) = n\cdot m$

$$X' = B(X,\mathbb{F}) \underset{\dim X<\infty}{=} L(X,\mathbb{F})$$

hence $\dim X' = \underbrace{\dim X}_{=n} \cdot \underbrace{\dim \mathbb{F}}_{=1} = n$

Since $\{f_1,\dots,f_n\} \subset X'$, $\dim X' = n$ and $f_1,\dots,f_n$ are n lin. indep. functionals $\Rightarrow$ $\{f_1,\dots,f_n\}$ is a basis for X'.

Theorem. (Riesz - Fréchet)
Let H be a Hilbert space. Consider $f \in H'$, then $\exists !\, y \in H$ such that

$$f(x) = (x, y), \qquad \forall x \in H \qquad (H).$$

Moreover $\|f\|_{H'} = \|y\|_H$.

Proof. Existence. Consider first the trivial case $f(x) = 0$, $\forall x \in H$, then we can write $f(x) = (x, \underset{\uparrow \text{zero vector}}{0})$, $\forall x \in H$ so that (H) is satisfied and $\|f\|_{H'} = \|0\|_{H'} = 0 = \|0\|_H$.

Consider now $f \in H' \setminus \{0\}$ hence $\operatorname{Ker} f \neq H$ so that $\operatorname{Ker} f \subset H$ (is a proper subspace), $\operatorname{Ker} f$ is closed since f is continuous so by the Orthogonal Decomposition Theorem

$$H = \operatorname{Ker} f \oplus \operatorname{Ker} f^{\perp}$$

so $\operatorname{Ker} f^{\perp} \neq \{0\}$. Consider $z \in \operatorname{Ker} f^{\perp}$ such that $f(z) \neq 0$, then $x \in H$ can be written as

$$x = \underbrace{x - \frac{f(x)}{f(z)} z}_{\in \operatorname{Ker} f} + \underbrace{\frac{f(x)}{f(z)} z}_{\in \operatorname{Ker} f^{\perp}} \quad \text{(subspace)}$$

In fact, $f\left(x - \frac{f(x)}{f(z)} z\right) = f(x) - \frac{f(x)}{\cancel{f(z)}} \cancel{f(z)} = 0$

Define $y = \alpha z$, with $\alpha \in \mathbb{F}$ to be determined such that condition (H) is satisfied

$$(x, y) = \Big(\underbrace{x - \frac{f(x)}{f(z)} z}_{\in \operatorname{Ker} f} + \underbrace{\frac{f(x)}{f(z)} z}_{\in \operatorname{Ker} f^{\perp}}, \underbrace{\alpha z}_{\in \operatorname{Ker} f^{\perp}}\Big) = \Big(\frac{f(x)}{f(z)} z, \alpha z\Big)$$

$$= \frac{f(x)}{f(z)} \bar{\alpha} (z, z) = \frac{f(x)}{f(z)} \bar{\alpha} \|z\|_H^2 = f(x)$$

in order to have (H) satisfied

$$\Rightarrow \frac{\bar{\alpha} \|z\|_H^2}{f(z)} = 1 \iff \bar{\alpha} = \frac{f(z)}{\|z\|^2}$$

$$\iff \alpha = \frac{\overline{f(z)}}{\|z\|^2}$$

hence $y = \alpha z$ satisfies (H).

Uniqueness. Assume $\exists\, y_1, y_2 \in H$:

$$f(x) = (x, y_1) = (x, y_2), \quad \forall x \in H$$

Hence $(x, y_1 - y_2) = 0, \quad \forall x \in H$

$$y_1 - y_2 \in H^{\perp} = \{0\} \Rightarrow y_1 = y_2$$

Let us show the equality : $\|f\|_{H'} = \|y\|_H, \; y \neq 0$

$$|f(x)| = |(x, y)| \overset{\text{Cauchy-Schwarz ineq.}}{\leq} \|x\|_H \|y\|_H, \quad \forall x \in H$$

hence $\|f\|_{H'} \leq \|y\|_X$

choose $x = \frac{y}{\|y\|}$, $\underset{|f(x)|}{\overset{\shortparallel}{f(x)}} = \left(\frac{y}{\|y\|}, y\right) = \frac{1}{\|y\|} \|y\|^2 = \|y\|$

$$\|f\|_{H'} = \sup\{\underbrace{|f(x)|}_{\|y\|_H^{\in}}, \; \forall x : \|x\|_H = 1\} \geq \|y\|_H$$

hence $\|f\|_{H'} = \|y\|_H$.