Theorem (Riesz - Fréchet)

H Hilbert space then $\forall f \in H'$ $\exists!\, y \in H$:

$$f(x) = (x, y), \qquad \forall x \in H \qquad (H)$$

and $\|f\|_{H'} = \|y\|_H$.

Remark. The previous theorem gives a representation of all elements of H'. Moreover, H' can be identified with H as we can see below.

Define $f_y(x) := (x, y)$, $\forall x \in H$

Theorem. H Hilbert space. Consider the mapping $T_H : H \to H'$, $T_H y = f_y$, $y \in H$. Then T_H is a bijection and $\forall \alpha, \beta \in \mathbb{F}$, $\forall y, z \in H$,

1) T_H is antilinear (conjugate-linear):

$$T_H(\alpha y + \beta z) = \bar{\alpha}\, T_H y + \bar{\beta}\, T_H z$$

2) T_H is an isometry : $\|T_H y\|_{H'} = \|y\|_H$, $\forall y \in H$.

Proof. From the Riesz - Fréchet theorem it follows that T_H is a bijection and T_H is an isometry. Let us show that T_H is antilinear

$$T_H(\alpha y + \beta z) = f_{\alpha y + \beta z}$$

$\forall x \in H$

$$f_{\alpha y + \beta z}(x) = (x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z)$$
$$= \bar{\alpha} f_y(x) + \bar{\beta} f_z(x)$$

hence $T_H(\alpha y + \beta z) = \bar{\alpha}\, T_H y + \bar{\beta}\, T_H z$. ■

Since T_H is an isometry we can define on H' the following inner product :

$$\underbrace{(f_y, f_z)_{H'}}_{\overset{\shortparallel}{(T_H y, T_H z)_{H'}}} = (y, z)_H$$

(check as exercise)

On H' we have the induced norm $\|f_y\| = \sqrt{(f_y, f_y)}$ which is $\|f_y\|_{H'} = \|y\|_H$, H is a Hilbert space so complete w.r.t. $\|\cdot\|_H \Rightarrow H'$ is complete w.r.t. $\|\cdot\|_{H'}$. So H' is a Hilbert space.

By abuse of notation: $H' = H$

(Observe: $T_H : H \to H'$ is an antilinear isometric isomorphism)

Examples. 1) $H = \ell^2$, then $T_H : \ell^2 \to (\ell^2)'$ is defined by $T_H y = f_y$ with $f_y(x) = (x, y)_{\ell^2} = \sum_{n=1}^{\infty} x_n \overline{y_n}$

2) $(X, \mathcal{M}, \mu)$ measure space and $H = L^2(X)$, then $T_H g = \varphi_g$ with $\varphi_g(f) = (f, g)_{L^2} = \int_X f(x)\overline{g(x)}\,d\mu$

Now we study the duals of ℓ^p, $1 \le p \le \infty$.

Theorem. Assume $1 \le p < \infty$. Set $1 \le q \le \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad (q \text{ conjugate exponent of } p)$$

Then,

1) If $a = (a_n) \in \ell^q$, then $\forall x = (x_n) \in \ell^p$ the sequence $(a_n x_n) \in \ell^1$. Define:

(A) $$f_a(x) = \sum_{n=1}^{\infty} a_n x_n, \qquad \forall x = (x_n) \in \ell^p.$$

Then $f_a \in (\ell^p)'$ and $\|f_a\|_{(\ell^p)'} = \|a\|_{\ell^q}$.

2) If $f \in (\ell^p)'$ then there exists a <u>unique</u> $a \in \ell^q$ such that $f = f_a$ defined in (A).

3) By items 1) and 2) the mapping $T_p : \ell^q \to (\ell^p)'$, defined by $T_p(a) = f_a$, $\forall a \in \ell^q$ is a linear isometric isomorphism.

Let us show that f_a is well defined and $f_a \in (\ell^p)'$: $\forall x = (x_n) \in \ell^p$

$$|f_a(x)| \overset{(A)}{=} \left| \sum_{n=1}^{\infty} a_n x_n \right| \leq \underbrace{\sum_{n=1}^{\infty} |a_n x_n|}_{= \|(a_n x_n)\|_{\ell^1}}$$

by Hölder's ineq.

$$\leq \|a\|_{\ell^q} \|x\|_{\ell^p} \quad \text{because } \frac{1}{q} + \frac{1}{p} = 1$$

So f_a is well defined and $|f_a(x)| \leq \|a\|_{\ell^q} \|x\|_{\ell^p}$, $\forall x \in \ell^p$ hence $\|f_a\|_{(\ell^p)'} \leq \|a\|_{\ell^q}$

$$f_a(\alpha x + \beta y) = \sum_{n=1}^{\infty} a_n (\alpha x_n + \beta y_n) \overset{\sum \text{ uncond. conv. since abs. conv}}{=} \alpha \sum_{n=1}^{\infty} a_n x_n + \beta \sum_{n=1}^{\infty} a_n y_n$$

$$= \alpha f_a(x) + \beta f_a(y).$$

Since T_p is an isometric isomorphism we can write, by abuse of notation, $(\ell^p)' = \ell^q$, $1 \leq p < \infty$. In particular, $(\ell^1)' = \ell^\infty$

$$\left(\frac{1}{1} + \frac{1}{\infty} = 1 \right)$$

Case $p = \infty$. The mapping $T_1 : \ell^1 \to (\ell^\infty)'$: $a \mapsto f_a$ defined in (A) is well defined in fact $f_a \in (\ell^\infty)'$, thanks to Hölder's ineq. as before we get $\|f_a\|_{(\ell^\infty)'} \leq \|a\|_{\ell^1}$.

It can be shown: $\|f_a\|_{(\ell^\infty)'} = \|a\|_{\ell^1}$, so the mapping T_1 is an isometry but it can be shown that T_1 is not <u>onto</u>.

So in this case, by abuse of notation, we can only write $\ell^1 \subset (\ell^\infty)'$

Lemma. The mapping $T: \ell^1 \to (c_0)'$, such that $Ta = f_a$, with $f_a(x) = \sum_{n=1}^{\infty} a_n x_n$, $\forall\, x = (x_n) \in c_0$ is an isometric isomorphism.

Similar results hold for $L^p(X)$, $1 \le p < \infty$

Given $(X, \mathcal{M}, \mu)$ measure space with μ σ-finite ($X = \bigcup_{n=1}^{\infty} E_n$, $E_n \in \mathcal{M} : \mu(E_n) < \infty$), for $1 \le p < \infty$, $1 \le q \le \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

then the mapping $T: L^q(X) \to (L^p(X))'$,

$g \mapsto \varphi_g$ with $\varphi_g(f) = \int_X f(x) g(x)\, d\mu$

is an isometric isomorphism.

Notation: $q = p'$, $\frac{1}{p'} + \frac{1}{p} = 1$

So, for $1 \le p < \infty$, we have:

$$(L^p(X))' = L^{p'}(X)$$

$$(\ell^p)' = \ell^{p'}$$

$$L^1(X) \subset (L^\infty(X))'$$

$$\ell^1 \subset (\ell^\infty)'$$

$$c_0' = \ell^1$$

Recall the "Density Principle" for functionals

X normed space, $W \subset X$ dense subspace of X

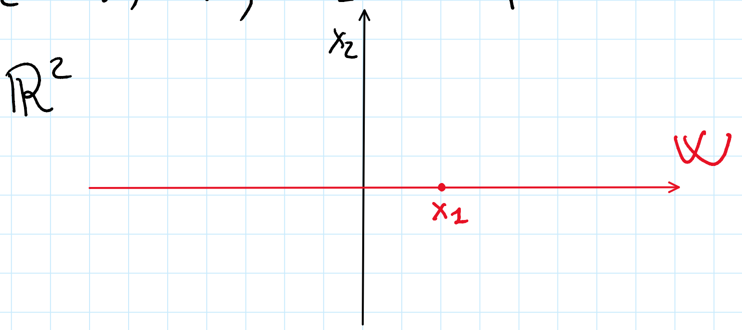
$f_W \in W'$ then $\exists!\, f_X \in X'$ extension of f_W:

1) $f_X(w) = f_W(w), \quad \forall w \in W$

2) $\|f_X\|_{X'} = \|f_W\|_{W'}$.

Exercise. Consider $H = \mathbb{R}^2$ and the subspace

$W = \{(x_1, 0),\ x_1 \in \mathbb{R}\} \simeq \mathbb{R}$



Consider $f_W : W \to \mathbb{R}$, defined by

$f_W(x_1) = 2x_1$.

1) Show that $f_W \in W'$

2) Compute $\|f_W\|_{W'}$

3) Construct an extension f_X of f_W on $\mathbb{R}^2$

$\left(\|f_X\|_{(\mathbb{R}^2)'} = \|f_W\|_{W'}, \quad f_W(x_1) = f_X(x_1, 0), \ \forall x_1 \in \mathbb{R} \right)$

Solution. 1) f_W is linear: $f_W(\alpha x_1 + \beta y_1) = 2(\alpha x_1 + \beta y_1)$

$= \alpha\, 2 x_1 + \beta\, 2 y_1$

$= \alpha f_W(x_1) + \beta f_W(y_1)$

$\dim W = \dim \mathbb{R} = 1$ hence any linear functional on W is also bounded $\Rightarrow f_W \in W'$.

2) $W = \mathbb{R}$ is a Hilbert space, so

$f_W(x_1) = 2x_1 = (x_1, 2)$

So by Riesz-Fréchet theorem, $\|f_W\|_{W'} = |2| = 2$

3) $\mathbb{R}^2$ is a Hilbert space, by Riesz-Fréchet theorem, any $f \in (\mathbb{R}^2)'$ is of the form

$$f(x_1, x_2) = ((x_1, x_2), (\alpha, \beta)), \quad \forall (x_1, x_2) \in \mathbb{R}^2$$

We want an extension of $f_W(x_1) = (x_1, 2) = 2x_1$

$f(x_1, x_2) = \alpha x_1 + \beta x_2$

In order to have $f_X(x_1, x_2) = f_W(x_1)$, $\forall x_1 \in \mathbb{R}$

$\Rightarrow \alpha = 2$.

Since $\|f_X\| = \|f_W\| = 2$ we must have

by Riesz-Fréchet $2 = \|f_X\|_{(\mathbb{R}^2)'} = \|(2, \beta)\|_{\mathbb{R}^2} = \sqrt{4+\beta^2}$

hence $2 = \sqrt{4+\beta^2} \Leftrightarrow 4 = 4 + \beta^2 \Leftrightarrow \beta = 0$

So $f_X(x_1, x_2) = ((x_1, x_2), (2, 0))$, $\forall (x_1, x_2) \in \mathbb{R}^2$

The Hahn-Banach Theorem for Normed Spaces

X normed space, $W \subset X$ subspace of X. For any $f_W \in W'$ there exists an extension $f_X \in X'$ of f_W ($f_X(w) = f_W(w)$, $\forall w \in W$, $\|f_X\|_{X'} = \|f_W\|_{W'}$)

Consequences :

Theorem 1. X normed space, $X \neq \{0\}$. Then

1) $\forall x_0 \in X$ $\exists f \in X' : \|f\|_{X'} = 1$ and $f(x_0) = \|x_0\|_X$.

2) $\|x_0\|_X = \sup\{ |f(x_0)| : f \in X', \|f\|_{X'} = 1 \}$

3) $\forall x, y \in X$, $x \neq y$, $\exists f \in X' : f(x) \neq f(y)$.

Proof. 1) Consider $x_0 \neq 0$, define

$W = Sp\{x_0\} = \{\lambda x_0, \lambda \in \mathbb{F}\}$

$\xrightarrow{\hspace{5cm}} W$

Define $\quad f_W(\lambda x_0) = \lambda \|x_0\|_X \, , \quad \forall \lambda \in \mathbb{F}$

by construction, f_W is linear on W and $f_W(x_0) = \|x_0\|_X$

$$|f_W(\lambda x_0)| = |\lambda \|x_0\|_X| = |\lambda| \, \|x_0\|_X = \|\lambda x_0\|_X$$

and $\quad \|f_W\|_{W'} = 1$.

Then by Hahn-Banach theorem $\exists f_X \in X'$:

$$\|f_X\|_{X'} = \|f_W\|_{W'} = 1 \quad \text{and} \quad f_X(x_0) = f_W(x_0) = \|x_0\|_X$$

For $x_0 = 0$, any functional $f \in X'$: $\|f\|_{X'} = 1$ satisfies $f(0) = 0 = \|0\|_X$.

2) $\quad \forall f \in X' : \|f\|_{X'} = 1$

$$|f(x_0)| \le \|f\|_{X'} \|x_0\|_X = \|x_0\|_X$$

So $\|x_0\|_X$ is an upper bound for the set

$$\{ |f(x_0)| : f \in X', \ \|f\|_{X'} = 1 \}$$

Moreover, by item 1) $\exists f \in X'$, $\|f\|_{X'} = 1$ such that $f(x_0) = |f(x_0)| = \|x_0\|_X$

So $\|x_0\|_X$ is a maximum and we have

$$\sup \{ |f(x_0)| : f \in X' : \|f\|_{X'} = 1 \} = \max \{ |f(x_0)| : f \in X', \|f\|_{X'} = 1 \}$$
$$= \|x_0\|_X$$

3) If $x \neq y \Rightarrow x - y \neq 0$ by item 1)

$\exists f \in X' : \|f\|_{X'} = 1$ and $f(x-y) = \|x-y\|_X \neq 0$

f lin. $\quad f(x-y) = f(x) - f(y)$

hence $f(x) - f(y) \neq 0$.

Remark 1) The previous theorem show another way to compute $\|x\|$, $\forall x \in X$. In fact, by item 2) we have

$$\|x\|_X = \sup_{f \in X', \|f\|_{X'} = 1} |f(x)| = \max_{f \in X', \|f\|_{X'} = 1} |f(x)|$$

2) For $f \in X'$, we have $\|f\|_{X'} = \sup_{x \in X : \|x\|_X = 1} |f(x)|$

but in this case in general $\|f\|_{X'}$ is a supremum and not a maximum.

3) If $X = H$ by the Riesz-Fréchet thrm $\|f\|_{H'}$ is a maximum.