Exercises n. 1-2-4 HMW3

Another consequence of Hahn-Banach Theorem:

Theorem 1 If X' is separable then X is separable. (X normed space).

Remark. The converse is not true in general.

Property. For $1 \le p < \infty$ ℓ^p is separable.

Sketch of proof. Recall $\forall x = (x_n) \in \ell^p$, if we consider $\{\delta_n\}_{n \in \mathbb{N}_+} \subset \ell^p$, $\delta_n = (0, \dots, \underbrace{1}_{n\text{-th entry}}, 0, \dots, 0, \dots)$

Hence $x = \sum_{n=1}^{\infty} x_n \delta_n$

$\forall k \in \mathbb{N}_+$, $S_k = \{ \sum_{n=1}^{k} \alpha_n \delta_n$, where $\alpha_n \in \mathbb{Q}$ if $\mathbb{F} = \mathbb{R}$ or $\alpha_n = r_n + i q_n$, $r_n, q_n \in \mathbb{Q} \}$

$S_k \simeq \mathbb{Q}^k$ if $\mathbb{F} = \mathbb{R}$ or $S_k \simeq \mathbb{Q}^{2k}$ if $\mathbb{F} = \mathbb{C}$

S_k is countable

$S := \bigcup_{k \in \mathbb{N}_+} S_k$ S is countable because countably union of countable sets.

S is dense in ℓ^p.

Property. ℓ^∞ is not separable.

Proof. Consider an arbitrary sequence $\{x^k\}_{k \in \mathbb{N}_+} \subset \ell^\infty$

$x^k = (x_1^k, x_2^k, \dots, x_n^k, \dots) \in \ell^\infty$

Consider the sequence $z = (z_n) \in \ell^\infty$ defined as follows

$$z_n = \begin{cases} x_n^n + 1, & \text{if } |x_n^n| \le 1 \\ 0, & \text{if } |x_n^n| > 1 \end{cases}$$

$$\|z - x^k\|_{\ell^\infty} = \sup_{1 \le n \le \infty} |z_n - x_n^k|$$

$$|z_n - x_n^k| \overset{n=k}{=} \begin{cases} |x_k^k + 1 - x_k^k| & |x_k^k| \le 1 \\ |0 - x_k^k| & |x_k^k| > 1 \end{cases}$$

$$= \begin{cases} 1, & |x_k^k| \le 1 \\ |x_k^k|, & |x_k^k| > 1 \end{cases}$$

$$\|z - x^k\|_{\ell^\infty} = \sup_{1 \le n \le \infty} |z_n - x_n^k| \geq |z_k - x_k^k| \geq 1$$

$$\Rightarrow \quad z \notin \overline{\{x^k\}}$$

$z \in \ell^\infty$, since $\|z\|_{\ell^\infty} = \sup_{1 \le n \le \infty} |z_n| \le 2$

$\Rightarrow$ ℓ^∞ is not separable. ∎

Remarks. 1) ℓ^1 is separable but $(\ell^1)' = \ell^\infty$ $\left(\frac{1}{1} + \frac{1}{\infty} = 1\right)$ is not separable. (This shows that the converse of Theorem 1 is not true in general).

2) Since ℓ^∞ is not separable, then by Theorem 1 we have $(\ell^\infty)'$ is not separable hence $(\ell^\infty)'$ is not isomorphic to ℓ^1 since separability is preserved by isomorphisms.

3) H Hilbert space, $H' = H$ hence H is separable $\Leftrightarrow$ H' is separable.

Bidual Spaces

Consider X normed space. Recall X' is a Banach space. So we can consider $(X')'$.

Definition. We define $X'' := (X')'$ and we call X'' the bidual of X.

(Another notation: $X^{**} = X''$)

Lemma X normed space. $x \in X$, define $\delta_x : X' \to \mathbb{F}$ by $\delta_x(f) = f(x)$, $\forall f \in X'$.

Then $\delta_x \in X''$ and $\|\delta_x\|_{X''} = \|x\|_X$.

Proof. δ_x is linear: $\forall \alpha, \beta \in \mathbb{F}$, $\forall f_1, f_2 \in X'$,

$$\delta_x(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x)$$
$$= \alpha \delta_x(f_1) + \beta \delta_x(f_2)$$

δ_x is bounded: $\forall f \in X'$

$$|\delta_x f| = |f(x)| \overset{f \text{ bounded}}{\leq} \|f\|_{X'} \|x\|_X$$

$\Rightarrow$ $\delta_x \in X''$ and $\|\delta_x\|_{X''} \leq \|x\|_X$

By the first consequence of Hahn-Banach Theorem, $\exists f \in X'$: $\|f\|_{X'} = 1$ and $f(x) = \|x\|_X$

$$\|\delta_x\|_{X''} = \sup \{ |\delta_x f| , f \in X' : \|f\|_{X'} = 1 \} \geq \|x\|_X$$

hence $\|\delta_x\|_{X''} \geq \|x\|_X$ $\Rightarrow$ $\|\delta_x\|_{X''} = \|x\|_X$.

Definition. X normed space. We define $J : X \to X''$ by $Jx = \delta_x$.

Lemma. The mapping $J : X \to X''$ is an isometry. Hence $J : X \to J(X)$ is an isometric isomorphism.

Proof. J is linear: $\forall \alpha, \beta \in \mathbb{F}$, $\forall x_1, x_2 \in X$

$$J(\alpha x_1 + \beta x_2) = \delta_{\alpha x_1 + \beta x_2}$$

$$\delta_{\alpha x_1 + \beta x_2}(f) = f(\alpha x_1 + \beta x_2) \overset{f \text{ lin.}}{=} \alpha f(x_1) + \beta f(x_2)$$
$$= \alpha \delta_{x_1}(f) + \beta \delta_{x_2}(f)$$
$$= (\alpha \delta_{x_1} + \beta \delta_{x_2})(f),$$

$\forall f \in X' \Rightarrow \delta_{\alpha x_1 + \beta x_2} = \alpha \delta_{x_1} + \beta \delta_{x_2}$

that is $J(\alpha x_1 + \beta x_2) = \alpha J x_1 + \beta J x_2$

J is an isometry by the previous lemma.

Since an isometry is one-to-one, then $J : X \to J(X)$ is an isomorphism, since $\|x\|_X = \|\delta_x\|_{J(X)}$ so we have the continuity of the inverse mapping.

Definition. If $J(X) = X''$ then we say that X is *reflexive*.

Observe that X'' is a Banach space hence if X is not complete then X is not reflexive since completeness is preserved by isomorphisms.

However, not every Banach space is reflexive.

Examples of reflexive spaces

1) Finite-dimensional vector spaces (ex. $\mathbb{R}^n$, $\mathbb{C}^n$)
2) Hilbert spaces

3) For $1 < p < \infty$, $L^p(X)$, ℓ^p are reflexive

$$(\ell^p)' = \ell^{p'} \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

$$(\ell^{p'})' = \ell^p \qquad \frac{1}{p'} + \frac{1}{p} = 1$$

$$\Rightarrow \ell^p = (\ell^p)''$$

Examples of not reflexive spaces

1) $L^1(X)$, ℓ^1, $L^\infty(X)$, ℓ^∞

2) $(\mathcal{C}_\#([a,b]), \|\cdot\|_\infty)$ is not reflexive

WEAK AND WEAK-$*$ CONVERGENCE

We limit to study the case X Banach space.

Definition. X Banach space. $\{x_n\} \subset X$ is said **weakly** convergent to $x \in X$, we write $x_n \rightharpoonup x$, $n \to \infty$, if $f(x_n) \to f(x)$, $n \to \infty$, $\forall f \in X'$.

(Equivalently, $\lim_{n \to \infty} f(x_n) = f(x)$, $\forall f \in X'$.)

Lemma. If $x_n \to x \in X$ ($\Leftrightarrow \|x_n - x\|_X \to 0$, $n \to \infty$) then $x_n \rightharpoonup x$.

Proof. $\forall f \in X'$,

$$|f(x_n) - f(x)| \overset{f \text{ lin}}{=} |f(x_n - x)| \overset{f \text{ bounded}}{\leq} \|f\|_{X'} \|x_n - x\|_X \to 0,$$

$n \to \infty$ hence by the comparison theorem,

$$\lim_{n \to \infty} |f(x_n) - f(x)| = 0 .$$

Example 1. H Hilbert space. Then $\{x_n\} \subset H$,

$$x_n \rightharpoonup x \in H \Leftrightarrow \forall y \in H, \ (x_n, y) \to (x, y)$$

Since by Riesz-Fréchet Theorem any $f \in H'$

is represented by $f(x) = f_y(x) = (x, y)$

Example 2. $H = \ell^2$, $\{\delta_n\} \subset \ell^2$ is an o.n.b.

$\|\delta_n\| = 1$, $\forall n$ $\quad \delta_n \not\to 0$

Let us show $\delta_n \rightharpoonup 0$.

By Example 1, we have to show $\forall y = (y_n) \in \ell^2$

$$\underset{\overline{y_n}}{\overset{\parallel}{(\delta_n, y)_{\ell^2}}} \longrightarrow (0, y)_{\ell^2} = 0$$

$\overline{y_n} \to 0$, $n \to \infty$ because $y = (y_n) \in \ell^2 \Rightarrow$

$\sum_{n=1}^{\infty} |y_n|^2 < \infty \Rightarrow |y_n|^2 \to 0 \Rightarrow |y_n| \to 0$

$\Leftrightarrow |\overline{y_n}| \to 0 \Leftrightarrow \overline{y_n} \to 0$

Example 3. $X = \ell^p$, $1 \le p < \infty$, we have $(\ell^p)' = \ell^{p'}$

$\frac{1}{p} + \frac{1}{p'} = 1$. Hence $\{x^k\} \subset \ell^p$, $x^k \rightharpoonup x \in \ell^p$

$\Leftrightarrow \forall a = (a_n) \in \ell^{p'}$ $\quad f_a(x^k) \to f_a(x)$

$$\Leftrightarrow \sum_{n=1}^{\infty} a_n x_n^k \xrightarrow{k \to \infty} \sum_{n=1}^{\infty} a_n x_n \text{ in } \mathbb{F}.$$

Example 4. $L^p(X)$, $1 \le p < \infty$, hence $f_n \rightharpoonup f$

$\Leftrightarrow \forall g \in L^{p'}(X)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\varphi_g(f_n) \to \varphi_g(f)$

$$\Leftrightarrow \int_X f_n g \, d\mu \xrightarrow{n \to \infty} \int_X f g \, d\mu \text{ in } \mathbb{F}.$$

Property. The weak limit is unique: if $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$ in X $\Rightarrow x = y$.

Proof. $\forall f \in X'$ we have $f(x_n) \to f(x)$ and $f(x_n) \to f(y)$ as $n \to \infty$, since the limit is unique : $f(x) = f(y)$.

Consider $f \in X'$ such that $f(x-y) = \|x-y\|_X$

$f(x) = f(y)$ but since f is linear:

$$\underbrace{f(x-y)}_{= \|x-y\|_X} = f(x) - f(y) = 0$$

hence $\|x-y\|_X = 0 \iff x-y=0 \iff x=y$.

Property. If $x_n \rightharpoonup x$ then $\{x_n\}$ is bounded.

Theorem "Characterization of weak convergence"

X Banach space. Then the following conditions are equivalent:

1) $x_n \rightharpoonup x \in X$

2) $\{x_n\}$ is bounded and $f(x_n) \longrightarrow f(x)$, $n \to \infty$
$\forall f \in S \mid S \subseteq X'$ with $\overline{Sp\, S} = X'$.