X Banach space. Types of convergence in X'

1) norm-convergence:

$$f_n \to f \in X' \iff \|f_n - f\|_{X'} \to 0, \quad n \to \infty$$

2) weak convergence:

$$f_n \rightharpoonup f \text{ in } X' \iff \forall F \in X'', \quad F(f_n) \to F(f), \; n \to \infty$$

3) weak-$*$ convergence:

$$f_n \xrightarrow{*} f \text{ in } X' \iff \forall x \in X \quad f_n(x) \to f(x), \; n \to \infty$$

Equivalently, $\forall x \in X \quad \delta_x f_n \to \delta_x f, \quad n \to \infty$

Hence it follows from the definition that if

$$f_n \rightharpoonup f \text{ in } X' \Rightarrow f_n \xrightarrow{*} f \text{ in } X'$$

in fact, one can choose $F \in X''$ to be $F = \delta_x$

Remark. If X is reflexive $\iff J(X) = X''$ then the weak and the weak-$*$ convergence coincide in X'.

Example H Hilbert space, then $H = H' = H''$

H is reflexive and so weak and weak-$*$ convergence coincide.

Lemma. X Banach space. Then

1) Weak-$*$ limits in X' are unique

2) If $f_n \xrightarrow{*} f \in X'$ then $\{f_n\} \subset X'$ is bounded.

One of the main reasons to introduce weak and weak-$*$ convergence is the following result.

Theorem "Banach-Alaoglu Theorem"

X separable Banach space. $\{f_n\} \subset X'$ a bounded sequence in X'. Then $\{f_n\}$ admits a subsequence $\{f_{n_k}\}$ that is weak-$*$ convergent to $f \in X'$.

$\left(\Leftrightarrow \quad f_{n_k} \xrightarrow{*} f, \quad k \to \infty \right)$

Corollary. X separable Banach space. We define $\overline{B_0(1)} = \{ f \in X' : \|f\|_{X'} \le 1 \} \subset X'$.

Then any sequence $\{f_n\} \subset \overline{B_0(1)}$ has a subseq. $\{f_{n_k}\} \mid f_{n_k} \xrightarrow{*} f \in \overline{B_0(1)}$.

We say that $\overline{B_0(1)} \subset X'$ is weak-$*$ compact.

PROJECTIONS AND COMPLEMENTARY SUBSPACES

Recall the Orthogonal Decomposition Theorem. H Hilbert space $Y \subset H$ closed subspace then $\forall x \in H \; \exists! \, (y,z)$, $y \in Y$, $z \in Y^\perp$ and $x = y + z$. $H = Y \oplus Y^\perp$

Definition. X vector space, U, V subspaces of X. U and V are said complementary subspaces if any $x \in X$ has a unique decomposition

(D) $\qquad x = u_x + v_x, \qquad u_x \in U, \quad v_x \in V$

If moreover X is a normed space and the mappings $x \mapsto u_x$ and $x \mapsto v_x$ are continuous then U and V are said topologically complementary subspaces.

Definition. X vector space. A projection P on X is a linear mapping $P: X \to X$ such that $P^2 = P$.
(Recall $P^2 = P \circ P$).

Lemma. X vector space, $P: X \to X$ projection on X.
Then:

1) $x \in \operatorname{Im} P \iff Px = x$
2) The operator $I - P$ is a projection and

(F) $\operatorname{Ker} P = \operatorname{Im}(I-P)$, $\operatorname{Im} P = \operatorname{Ker}(I-P)$

Proof. 1) $\Rightarrow$ $x \in \operatorname{Im} P$ then $\exists y \in X : Py = x$

$$Px = P(Py) = P^2 y \overset{P^2=P}{=} Py = x$$

$\Leftarrow$ $Px = x$ $\Rightarrow$ trivially $x \in \operatorname{Im} P$.

2) I is a linear operator, P is linear $\Rightarrow$
$I-P$ is a linear operator and $I-P: X \to X$

$$(I-P)^2 = I - 2P + \underbrace{P^2}_{=P} = I-P$$

$\Rightarrow$ $I-P$ is a projection

Let us show $\operatorname{Ker} P = \operatorname{Im}(I-P)$

$y \in \operatorname{Im}(I-P)$ then $\exists x \in X : (I-P)x = y$

$$Py = P(I-P)x = Px - \underbrace{P^2}_{=P} x = Px - Px = 0$$

$\Rightarrow y \in \operatorname{Ker} P$ hence $\operatorname{Im}(I-P) \subseteq \operatorname{Ker} P$

Now we show $\operatorname{Ker} P \subseteq \operatorname{Im}(I-P)$

$y \in \operatorname{Ker} P \Rightarrow Py = 0$, $y = \underbrace{Iy}_{=y} - \underbrace{Py}_{=0} = (I-P)y$

$\Rightarrow y \in \operatorname{Im}(I-P)$.

So $\operatorname{Ker} P = \operatorname{Im}(I-P)$. The relation $\operatorname{Im} P = \operatorname{Ker}(I-P)$ can be proved similarly.

"Connection between projections and complementary subspaces"

Lemma. X vector space.

1) $U, V \subseteq X$ complementary subspaces. Define the operators $P_U : X \to U \subseteq X$, $P_V : X \to V \subseteq X$ as follows:

$P_U x = u_x$, $P_V x = v_x$ where $x = u_x + v_x$ in (D)

Then P_U and P_V are projections on X and $P_U + P_V = I$.

2) If P is a projection on X then the subspaces $\operatorname{Ker} P$ and $\operatorname{Im} P$ are complementary subspaces.

Proof. 1) Consider $P_U : X \to U \subseteq X$ by def. Let us show P_U is a projection.

- P_U is linear: $\forall \alpha, \beta \in \mathbb{F}$, $\forall x, y \in X$ then

$x = u_x + v_x$, $y = u_y + v_y$ (unique dec.)

$$P_U(\alpha x + \beta y) = P_U(\alpha(u_x + v_x) + \beta(u_y + v_y))$$
$$= P_U(\underbrace{(\alpha u_x + \beta u_y)}_{\in U} + \underbrace{(\alpha v_x + \beta v_y)}_{\in V}) = \alpha u_x + \beta u_y$$
$$= \alpha P_U x + \beta P_U y$$

$P_U^2 x = P_U(\underbrace{P_U x}_{= u_x}) = P_U u_x = u_x = P_U x$ $\quad u_x = u_x + 0$,

$\forall x \in X$ hence $P_U^2 = P_U$ that is P_U is a project.

Similarly one can show that P_V is a projection.

Finally $P_U + P_V = I$ comes from the decomposition $u_x + v_x = x$, $\forall x \in X$ (formula (D))

2) Assume P projection, $\forall x \in X$ define

$u_x := Px \in \operatorname{Im} P \qquad v_x := (I-P)x \in \underbrace{\operatorname{Im}(I-P)}_{= \operatorname{Ker} P}$

$x = Px + \underbrace{Ix - Px}_{=(I-P)x} = u_x + v_x, \quad u_x \in \operatorname{Im} P, \quad v_x \in \operatorname{Ker} P$

Let us show that the decomposition above is unique: assume $x = u_x + v_x = u'_x + v'_x$, $u_x, u'_x \in \operatorname{Im} P$, $v_x, v'_x \in \operatorname{Ker} P$. Then

$$u_x - u'_x = v'_x - v_x$$

Applying P to the left-hand side: $u_x \in \operatorname{Im} P$ $u'_x \in \operatorname{Im} P$ hence $Pu_x = u_x$, $Pu'_x = u'_x$

$$P(u_x - u'_x) = Pu_x - Pu'_x = u_x - u'_x$$

$v'_x \in \operatorname{Ker} P$ $v_x \in \operatorname{Ker} P$

$$P(v'_x - v_x) = \underbrace{Pv'_x}_{=0} - \underbrace{Pv_x}_{=0} = 0$$

$$u_x - u'_x = 0 \iff u_x = u'_x$$

hence $v'_x - v_x = u_x - u'_x = 0 \implies v'_x = v_x$.

Definition. The projection P_U is called the "projection of X onto U along V", similarly for P_V.

Lemma. X normed space. $U, V \subseteq X$ complementary subspaces. Then

1) U and V are topologically complementary $\iff$ P_U and P_V are bounded

2) If U and V are topologically complementary then U and V closed.

Proof. 1) It follows immediately from the definition of topologically complementary subspaces.

2) $U = \operatorname{Im} P_U \overset{P_U \text{ is a projection}}{=} \ker(I - P_U)$

$I \in B(X)$, P_U by assumption is continuous

$\iff P_U$ is bounded $\implies I - P_U \in B(X)$

hence $\operatorname{Ker}(I - P_U)$ is closed.

Similarly, $V = \operatorname{Ker}(I - P_V)$.

Lemma. X Banach space. $U, V \subseteq X$ closed complementary subspaces in X. Then U and V are topologically complementary.

Example H Hilbert space, $Y \subset H$ closed subspace then $H = Y \oplus Y^{\perp}$, hence $Y, Y^{\perp}$ are closed complementary subspaces and by the previous lemma Y and $Y^{\perp}$ are topologically complementary.

Ex. n. 3 HMW 3.