THE ADJOINT OF AN OPERATOR

From now on we consider $\mathbb{F} = \mathbb{C}$.

Theorem. Consider H and K be complex Hilbert spaces. Let T be in $B(H,K)$. Then there exists a <u>unique</u> operator $T^* \in B(K,H)$ such that

(A) $$(Tx, y)_K = (x, T^*y)_H, \quad \forall x \in H, \forall y \in K.$$

Proof. Consider $y \in K$. Define $f(x) = (Tx, y)_K$, $\forall x \in H$. Observe that $f : H \to \mathbb{C}$ well defined.

- f is linear: $\forall \alpha, \beta \in \mathbb{C}$, $\forall x_1, x_2 \in H$,

$$f(\alpha x_1 + \beta x_2) = (T(\alpha x_1 + \beta x_2), y)_K \overset{T \text{ lin}}{=} (\alpha T x_1 + \beta T x_2, y)_K$$
$$\overset{(\cdot,\cdot) \text{ lin}}{=} \alpha (Tx_1, y)_K + \beta (Tx_2, y)_K$$
$$= \alpha f(x_1) + \beta f(x_2)$$

- f is bounded:

$$|f(x)| = |(Tx, y)_K| \overset{\text{Cauchy-Schwarz}}{\leq} \|Tx\|_K \|y\|_K$$
$$\overset{T \in B(H,K)}{\leq} \|T\|_{B(H,K)} \|x\|_H \|y\|_K, \quad \forall x \in H$$

$$|f(x)| \leq \left(\|T\|_{B(H,K)} \|y\|_K \right) \|x\|_H, \quad \forall x \in H$$

hence $f \in H'$. By Riesz-Fréchet theorem, $\exists!\, z$ in H such that $f(x) = (x, z)_H$, $\forall x \in H$.

We define $T^* y = z$. Hence we have constructed a mapping $T^* : K \to H$ which satisfies the equality in (A).

- T^* is a linear operator:

$\forall \alpha, \beta \in \mathbb{C}$, $\forall y_1, y_2 \in K$, $\forall x \in H$

$$(x, T^*(\alpha y_1 + \beta y_2))_H \underset{(,\cdot)\text{ anti-linear}}{\overset{(A)}{=}} (Tx, \alpha y_1 + \beta y_2)_K$$

$$= \bar{\alpha}(Tx, y_1)_K + \bar{\beta}(Tx, y_2)_K$$

$$\overset{(A)}{=} \bar{\alpha}(x, T^* y_1)_H + \bar{\beta}(x, T^* y_2)_H$$

$$\underset{(,\cdot)\text{ anti-lin}}{=} (x, \alpha T^* y_1 + \beta T^* y_2)_H$$

$$(x, T^*(\alpha y_1 + \beta y_2) - (\alpha T^* y_1 + \beta T^* y_2))_H = 0, \ \forall x \in H$$

$$\Rightarrow T^*(\alpha y_1 + \beta y_2) - (\alpha T^* y_1 + \beta T^* y_2) \in H^\perp = \{0\}$$

$$\Rightarrow T^*(\alpha y_1 + \beta y_2) = \alpha T^* y_1 + \beta T^* y_2$$

- T^* is bounded

$$\|T^* y\|_H^2 = (T^* y, T^* y)_H \overset{(A)}{=} (TT^* y, y)_K$$

$$\underset{\text{Cauchy-Schwarz ineq.}}{\le} \|TT^* y\|_K \|y\|_K$$

$$\underset{T \in B(H,K)}{\le} \|T\|_{B(H,K)} \|T^* y\|_H \|y\|_K$$

If $\|T^* y\|_H = 0$ then $\|T^* y\|_H \le C \|y\|_H$ is trivially satisfied $\forall y \in K$.

If $\|T^* y\|_H \neq 0$, from the previous ineq.:

$$\|T^* y\|_H \le \|T\|_{B(H,K)} \|y\|_K, \quad \forall y \in K$$

Hence $T^* \in B(K, H)$ and $\|T^*\|_{B(K,H)} \le \|T\|_{B(H,K)}$.

- T^* is unique:

assume $\exists B_1, B_2 \in B(K, H)$ such that (A) is satisfied: $(Tx, y)_K = (x, B_1 y)_H = (x, B_2 y)_H$, $\forall x \in H$, $\forall y \in K$. Hence:

$$(x, B_1 y - B_2 y) = 0, \quad \forall x \in H$$

$$\Rightarrow B_1 y - B_2 y \in H^\perp = \{0\} \Rightarrow B_1 y = B_2 y,$$

$\forall y \in K$ hence $B_1 = B_2$. ■

Definition. The operator T^* constructed above is called **the adjoint of T**.

Remark. The uniqueness of T^* is very useful when finding the adjoint. In fact, if we find an operator B which satisfies:

$$(Tx, y)_K = (x, By)_{H}, \quad \forall x \in H, \ \forall y \in K$$

then if $B \in \mathcal{B}(K, H)$ we have $B = T^*$.

Example. $H = K = \mathbb{C}^2$. Consider the standard o.n.b. $\{e_1, e_2\}$ of $\mathbb{C}^2$. Then if $T \in \mathcal{B}(\mathbb{C}^2)$ then T can be represented by means of a matrix $M(T) = [a_{ij}]$ (2×2 matrix), $i, j = 1, 2$. Then $T^* \in \mathcal{B}(\mathbb{C}^2)$ and $M(T^*) = [\overline{a_{ji}}]$, $i, j = 1, 2$.

Solution. $T^* \in \mathcal{B}(\mathbb{C}^2)$ by the previous theorem

$\Rightarrow \quad M(T^*) = [b_{ij}], \quad i, j = 1, 2.$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad x_i, y_i \in \mathbb{C}, \quad i = 1, 2$

$$(Tx, y)_{\mathbb{C}^2} = (x, T^* y)_{\mathbb{C}^2}$$

$$\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)$$

$$\left(\begin{bmatrix} a_{11} x_1 + a_{12} x_2 \\ a_{21} x_1 + a_{22} x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} b_{11} y_1 + b_{12} y_2 \\ b_{21} y_1 + b_{22} y_2 \end{bmatrix} \right)$$

$$a_{11}x_1\overline{y_1} + a_{12}x_2\overline{y_1} + a_{21}x_1\overline{y_2} + a_{22}x_2\overline{y_2} = x_1\overline{b_{11}}\,\overline{y_1} + x_1\overline{b_{12}}\,\overline{y_2} + x_2\overline{b_{21}}\,\overline{y_1} + x_2\overline{b_{22}}\,\overline{y_2}$$

$\forall x_1, x_2, y_1, y_2 \in \mathbb{C}$

choose $x_1 = y_1 = 1$, $x_2 = y_2 = 0 \Rightarrow a_{11} = \overline{b_{11}} \Rightarrow b_{11} = \overline{a_{11}}$

$x_2 = y_1 = 1$, $x_1 = y_2 = 0 \Rightarrow a_{12} = \overline{b_{21}} \Rightarrow b_{21} = \overline{a_{12}}$

$x_1 = y_2 = 1$, $x_2 = y_1 = 0 \Rightarrow a_{21} = \overline{b_{12}} \Rightarrow b_{12} = \overline{a_{21}}$

$x_2 = y_2 = 1$, $x_1 = y_1 = 0 \Rightarrow a_{22} = \overline{b_{22}} \Rightarrow b_{22} = \overline{a_{22}}$

hence $[b_{ij}] = [\overline{a_{ji}}]$. ∎

Remark. More generally, if $\{e_1, \dots, e_m\}$ is the o.n.b. for $\mathbb{C}^m$, $\{e_1, \dots, e_n\}$ the o.n.b. for $\mathbb{C}^n$, then any $T \in \mathcal{B}(\mathbb{C}^n, \mathbb{C}^m)$ can be represented by $M(T) = [a_{ij}]$, $i = 1, \dots, m$, $j = 1, \dots, n$ then $T^* \in \mathcal{B}(\mathbb{C}^m, \mathbb{C}^n)$ and $M(T^*) = [\overline{a_{ji}}]$ $j = 1, \dots, n$, $i = 1, \dots, m$.

Definition. If $A = [a_{ij}]$ is a $m \times n$ matrix the matrix $A^* = [\overline{a_{ji}}]$ ($n \times m$ matrix) is called **the adjoint of A**.

Example. $H = K = L^2([0,1])$. Fix $h \in C([0,1])$

Consider $T_h f(x) = h(x) f(x)$ a.e. $x \in [0,1]$

$T_h \in \mathcal{B}(L^2([0,1]))$. Then $T_h^* = T_{\overline{h}}$.

Solution. $\forall f, g \in L^2([0,1])$, study (A):

$$(T_h f, g)_{L^2([0,1])} = (f, T_h^* g)_{L^2([0,1])}$$

$$(T_h f, g)_{L^2} = \int_0^1 T_h f(x)\, \overline{g(x)}\, dx = \int_0^1 h(x) f(x)\, \overline{g(x)}\, dx$$

$$= \int_0^1 f(x) \left(h(x) \overline{g(x)} \right) dx$$

$$= \int_0^1 f(x) \left(\overline{\overline{h(x)}\, g(x)} \right) dx$$

$$= \left(f, \overline{h} g \right)_{L^2}$$

$$T_h^* g = \overline{h} g, \quad \forall g \in L^2([0,1])$$

Observe that $\overline{h} \in C([0,1])$ hence $T_h^* = T_{\overline{h}}$.

Example. $H = K = L^2([a,b])$. $k \in L^2([a,b]\times[a,b])$

Consider $Tf(x) = \int_a^b k(x,y) f(y)\, dy$, the integral operator with kernel k. We know $T \in B(L^2([a,b]))$

Then $T^* \in B(L^2([a,b]))$ is an integral operator with kernel $\tilde{k}(x,y) = \overline{k(y,x)}$.

Solution. $\forall f, g \in L^2([a,b])$, we study:

$$(Tf, g)_{L^2} = (f, T^* g)_{L^2}$$

$$(Tf, g)_{L^2} = \int_a^b Tf(x) \overline{g(x)}\, dx$$

$$= \int_a^b \left(\int_a^b k(x,y) f(y)\, dy \right) \overline{g(x)}\, dx$$

let us check that $k(x,y) f(y) \overline{g(x)} \in L^1([a,b]\times[a,b])$

Hölder's in. with $p=q=2$

$$\int_a^b \int_a^b \left| k(x,y) f(y) \overline{g(x)} \right| dx\,dy \le \left(\int_a^b \int_a^b |k(x,y)|^2 dx\,dy \right)^{1/2} \cdot \left(\int_a^b \int_a^b |f(y)|^2 |g(x)|^2 dx\,dy \right)^{1/2}$$

Tonelli's Theorem

$$= \|k\|_{L^2([a,b]\times[a,b])} \|g\|_{L^2([0,1])} \|f\|_{L^2([0,1])} < \infty$$

So by Fubini's theorem

$$\int_a^b \left(\int_a^b K(x,y) f(y)\, dy \right) \overline{g(x)}\, dx = \int_a^b f(y) \int_a^b K(x,y) \overline{g(x)}\, dx\, dy$$
$$= \left(f, \overline{\int_a^b K(x,\cdot) \overline{g(x)}\, dx} \right)_{L^2}$$

$$T^* g(y) = \overline{\int_a^b K(x,y) \overline{g(x)}\, dx} = \int_a^b \overline{K(x,y)}\, g(x)\, dx,$$
$\forall g \in L^2([a,b])$.

So $\quad T^* g(x) = \int_a^b \overline{K(y,x)}\, g(y)\, dy$

that is T^* is an integral operator with kernel $\tilde{K}(x,y) = \overline{K(y,x)}$.

Example. $H = K = \ell^2$. Consider $S \in B(\ell^2)$ the unilater shift. $\forall x = (x_n) \in \ell^2$, $Sx = (0, x_1, x_2, \dots)$.

Then $S^* \in B(\ell^2)$ is given by, $\forall y = (y_n)_{n \in \mathbb{N}_+} \in \ell^2$

$$S^* y = (y_2, y_3, \dots, y_n, \dots).$$

Solution. Study (A):

$$(Sx, y)_{\ell^2} = (x, S^* y)_{\ell^2}$$

Write $z = S^* y$

$$(Sx, y) = \left((0, x_1, x_2, x_3, \dots), (y_1, y_2, y_3, \dots) \right)$$
$$= x_1 \overline{y_2} + x_2 \overline{y_3} + x_3 \overline{y_4} + \dots$$
$$\overset{(A)}{=} x_1 \overline{z_1} + x_2 \overline{z_2} + x_3 \overline{z_3} + \dots$$

$\forall x = (x_n) \in \ell^2$

choose $x_1 = 0$, $x_n = 0$, $\forall n \geq 2 \Rightarrow \overline{y_2} = \overline{z_1} \Leftrightarrow z_1 = y_2$

choose $x_2 = 1$, $x_n = 0$, $\forall n \neq 2 \Rightarrow \overline{y_3} = \overline{z_2} \Leftrightarrow z_2 = y_3$

proceeding this way we get $z_n = y_{n+1}$

hence $\quad z = S^* y = (y_2, y_3, \dots \quad)$.

Example. H complex Hilbert space. If I is the identity operator on H, then $I^* = I$.

Solution. $\forall x, y \in H$, rel. (A) becomes:

$$(\underbrace{Ix}_{=x}, y)_H = (x, I^* y)_H, \quad \forall x \in H$$

$$\Rightarrow \quad y = I^* y, \qquad \forall y \in H$$

hence $\quad I^* = I$.

Remark. If A is a $m \times n$ matrix and B is a $n \times \ell$ matrix, $\lambda, \mu \in \mathbb{C}$, then

1) $(AB)^T = B^T A^T$

2) $(AB)^* = B^* A^*$

If A, B are $m \times n$ matrices then

$$(\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^*.$$

Lemma. $H, K, \mathcal{L}$ complex Hilbert spaces, $\lambda, \mu \in \mathbb{C}$, $R, S \in \mathcal{B}(H, K)$, $T \in \mathcal{B}(K, \mathcal{L})$. Then

1) $(\lambda R + \mu S)^* = \bar{\lambda} R^* + \bar{\mu} S^*$

2) $(TR)^* = R^* T^*$

Proof. 1)

$$\begin{aligned}((\lambda R + \mu S)x, y)_K &= (\lambda Rx + \mu Sx, y)_K \\ &= \lambda (Rx, y)_K + \mu (Sx, y)_K \\ &= \lambda (x, R^* y)_H + \mu (x, S^* y)_H \\ &= (x, (\bar{\lambda} R^* + \bar{\mu} S^*) y)_H\end{aligned}$$

hence $\quad (\lambda R + \mu S)^* = \bar{\lambda} R^* + \bar{\mu} S^*$

2) $(TRx, y) = (Rx, T^* y) = (x, R^* T^* y)$

$\forall x \in H, \quad \forall y \in \mathcal{K}$, hence $(TR)^* = R^* T^*$.