Theorem. H, K complex Hilbert spaces, $T \in B(H,K)$.

Then,

(1) $(T^*)^* = T$

(2) $\|T^*\|_{B(K,H)} = \|T\|_{B(H,K)}$

(3) The function $f: B(H,K) \to B(K,H) : T \mapsto T^*$ is continuous.

(4) $\|T^*T\|_{B(H)} = \|T\|^2_{B(H,K)}$.

Proof. (1) $\forall x \in H$, $\forall y \in K$

$$(y, (T^*)^* x)_K \overset{(A)}{=} (T^* y, x)_H = \overline{(x, T^* y)_H} \overset{(A)}{=} \overline{(Tx, y)_K}$$
$$= (y, Tx)_K$$

$\forall y \in K \Rightarrow (T^*)^* x = Tx$, $\forall x \in H \Rightarrow (T^*)^* = T$.

(2) From the proof of the theorem defining the adjoint T^* we get $\|T^*\|_{B(K,H)} \le \|T\|_{B(H,K)}$ $(*)$

Now, using item (1), $\|T\|_{B(H,K)} \overset{(1)}{=} \|(T^*)^*\|_{B(H,K)} \le$

(*) with T^* in place of T

$\le \|T^*\|_{B(K,H)}$

$\Rightarrow \|T\|_{B(H,K)} = \|T^*\|_{B(K,H)}$.

(3) $\forall \varepsilon > 0$ choose $\delta = \varepsilon$, $\forall R, S \in B(H,K)$ / $\|R-S\|_{B(H,K)} < \delta$ then

$$\|f(R) - f(S)\|_{B(K,H)} = \|R^* - S^*\|_{B(K,H)} = \|(R-S)^*\|_{B(K,H)}$$
$$\overset{(2)}{=} \|R-S\|_{B(H,K)} < \varepsilon$$

(4) $$\|T^*T\|_{B(H)} \leq \|T^*\|_{B(K,H)}^{=\|T\|_{B(H,K)}} \|T\|_{B(H,K)} = \|T\|^2_{B(H,K)}$$

$$\|Tx\|_K^2 = (Tx, Tx)_K \overset{(A)}{=} (T^*Tx, x)_H \underset{\text{Cauchy-Schwarz ineq.}}{\leq} \|T^*Tx\|_H \|x\|_H \leq \|T^*T\|_{B(H)} \|x\|^2_H$$

$$\|Tx\|_K \leq \sqrt{\|T^*T\|_{B(H)}}\, \|x\|_H, \quad \forall x \in H$$

$$\Rightarrow \quad \|T\|_{B(H,K)} \leq \sqrt{\|T^*T\|_{B(H)}}$$

$$\Rightarrow \quad \|T^*T\|_{B(H)} = \|T\|^2_{B(H,K)}. \qquad \blacksquare$$

Lemma. H, K complex Hilbert spaces, $T \in B(H,K)$. Then,

(1) $\operatorname{Ker} T = (\operatorname{Im} T^*)^\perp$

(2) $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$

(3) $\overline{\operatorname{Im} T} = (\operatorname{Ker} T^*)^\perp$

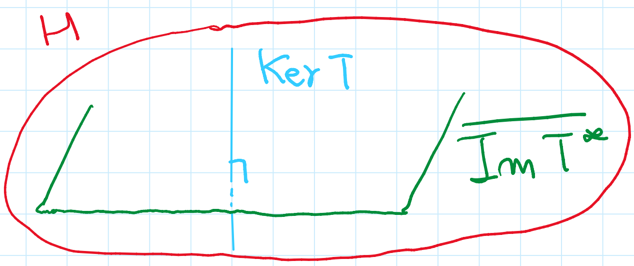
(4) $\overline{\operatorname{Im} T^*} = (\operatorname{Ker} T)^\perp$

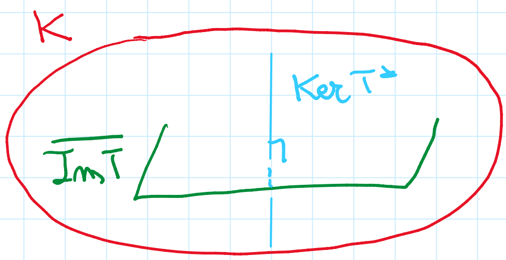
(5) $\operatorname{Ker} T^* = \{0\} \iff \operatorname{Im} T$ is dense in K.

Remark. In finite-dimensional Hilber spaces we have $\overline{\operatorname{Im} T} = \operatorname{Im} T$ since $\operatorname{Im} T$ is finite dimensional hence it is closed.

$$K = \operatorname{Ker} T^* \oplus \overline{\operatorname{Im} T}$$

$$H = \operatorname{Ker} T \oplus \overline{\operatorname{Im} T^*}$$





If you have to study $Tx = y$, you do not have to consider any $y \in K$ but only $y \in \overline{\operatorname{Im} T}$ that is to say $y \in (\operatorname{Ker} T^*)^\perp$

So it is important to know $\operatorname{Ker} T^*$.

Proof. (1) let us show $\operatorname{Ker} T \subseteq (\operatorname{Im} T^*)^\perp$

$x \in \operatorname{Ker} T$, $\forall z \in \operatorname{Im} T^*$ $\exists y \in K : T^* y = z$

$$(x, z)_H = (x, T^* y)_H \overset{(A)}{=} (\underbrace{Tx}_{\substack{=0 \\ x \in \operatorname{Ker} T}}, y)_K = 0$$

$$\Rightarrow \quad x \in (\operatorname{Im} T^*)^\perp$$

Now, we have to show $(\operatorname{Im} T^*)^\perp \subseteq \operatorname{Ker} T$

$v \in (\operatorname{Im} T^*)^\perp$ $\qquad T^* T v \in \operatorname{Im} T^*$

$$(Tv, Tv)_K \overset{(A)}{=} (\underbrace{T^* T v}_{\in \operatorname{Im} T^*}, \underset{\substack{\cap \\ (\operatorname{Im} T^*)^\perp}}{v})_H = 0$$

$$\| Tv \|_K^2 = 0 \Rightarrow Tv = 0 \Rightarrow v \in \operatorname{Ker} T.$$

Hence $\operatorname{Ker} T = (\operatorname{Im} T^*)^\perp$.

(2) Using item (1)

$$\operatorname{Ker} T^* \overset{(1)}{=} (\operatorname{Im} (T^*)^*)^\perp = (\operatorname{Im} T)^\perp$$

(3) Taking the orthogonal in the equality (2)

$$(\operatorname{Ker} T^*)^\perp = (\operatorname{Im} T)^{\perp\perp} = \overline{\operatorname{Im} T}$$

(4) It follows by taking the orthogonal complements of item (1).

(5) by item (2) $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$

so, assume $\operatorname{Ker} T^* = \{0\}$ then

$$(\operatorname{Im} T)^\perp = \{0\} \Rightarrow \overline{\operatorname{Im} T} = (\operatorname{Im} T)^{\perp\perp} = \{0\}^\perp = K$$

$\Rightarrow \operatorname{Im} T$ is dense in K.

Vice versa, if $\operatorname{Im} T$ is dense in $K \Leftrightarrow$
$\overline{\operatorname{Im} T} = K$ $\qquad K^\perp = \left(\overline{\operatorname{Im} T}\right)^\perp = (\operatorname{Im} T)^\perp = \operatorname{Ker} T^*$

$\qquad\qquad\qquad\qquad\qquad\quad \overset{\|}{\{0\}}$

$\Rightarrow \operatorname{Ker} T^* = \{0\}$.

Corollary. H, K complex Hilbert spaces and $T \in B(H,K)$ then T is invertible $\Leftrightarrow$ $\operatorname{Ker} T^* = \{0\}$ and $\exists \alpha > 0 : \quad \|Tx\|_K \geq \alpha \|x\|_H, \quad \forall x \in H$.

Proof. Use the characterization for invertible operators and replace "$\operatorname{Im} T$ dense in K" by "$\operatorname{Ker} T^* = \{0\}$".

Example. $S : \ell^2 \to \ell^2$ unilateral shift, $\forall x = (x_n) \in \ell^2$

$Sx = (0, x_1, x_2, \dots, x_n, \dots)$, $\qquad S \in B(\ell^2)$.

$\operatorname{Ker} S$? $\qquad Sx = 0 \Leftrightarrow (0, x_1, x_2, \dots) = (0, 0, \dots, 0, \dots)$

$\Rightarrow x_n = 0, \ \forall n \quad \Rightarrow x = (0) = 0$

$\operatorname{Ker} S = \{0\}$ $\qquad \operatorname{Im} S^*$?

$S^* : (y_1, y_2, \dots, y_n, \dots) \to (y_2, y_3, \dots, y_n, \dots)$

$\operatorname{Im} S^* = \ell^2$, in fact $\forall z = (z_n) \in \ell^2 \ \exists y = (y_n) \in \ell^2$ such that $S^* y = z$

$(y_2, y_3, \dots, y_n, \dots) = (z_1, z_2, \dots, z_n, \dots)$

$y_2 = z_1, \quad y_3 = z_2, \quad \dots, \quad y_{n+1} = z_n, \quad \dots$

So a counterimage of z is $y = (0, z_1, z_2, \dots) \in \ell^2$

or any sequence of the type $(a, z_1, z_2, \dots)$, with $a \in \mathbb{C}$. So S^* is onto.

$\operatorname{Ker} S^* = \{ y \in \ell^2 : S^* y = 0 \} = \{ (a, 0, 0, \dots, 0, \dots), a \in \mathbb{C} \}$

$= \operatorname{Span} \delta_1 \qquad \delta_1 = (1, 0, 0, 0, \dots 0 \dots)$

S^* is not injective ($\Rightarrow$ S^* is not invertible)

$\overline{\operatorname{Im} S} = (\operatorname{Ker} S^*)^\perp = \{ y \in \ell^2 : y = (y_n),\ y_1 = 0 \}$

$= \operatorname{Im} S$

$\overline{\operatorname{Im} S} = \operatorname{Im} S$ so $\operatorname{Im} S$ is a proper closed subspace of ℓ^2.

Lemma. H complex Hilbert space, $T \in B(H)$.

Then T is invertible $\Leftrightarrow$ T^* is invertible

with $(T^*)^{-1} = (T^{-1})^*$.

Proof. T is invertible $\Leftrightarrow$ $\exists\, T^{-1} \in B(H)$:

$$T^{-1} T = I = T T^{-1}$$

taking the adjoints,

$$(T^{-1} T)^* = I^* = (T T^{-1})^*$$

$$T^* (T^{-1})^* = \overset{\shortparallel}{I} = (T^{-1})^* T^*$$

hence $\Leftrightarrow$ T^* is invertible and

$$(T^*)^{-1} = (T^{-1})^*.$$

SELF-ADJOINT, NORMAL AND UNITARY OPERATORS

Definition. H complex Hilbert space, $T \in B(H)$.

- T is self-adjoint $\Leftrightarrow$ $T^* = T$
- T is normal $\Leftrightarrow$ $T T^* = T^* T$

- T is unitary $\iff T^*T = I = TT^*$
  (T is invertible and $T^{-1} = T^*$)

Examples. 1) $I^* = I \implies I$ is self-adjoint and unitary

2) For $h \in \mathcal{C}_{\mathbb{F}}([0,1])$, consider $T_h \in B(L^2([0,1]))$

$\forall f \in L^2([0,1])$, $T_h f(x) = h(x) f(x)$

We know: $T_h^* = T_{\bar{h}}$

$$T_h^* T_h f(x) = T_{\bar{h}} T_h f(x) = \overline{h(x)}\, T_h f(x) = \overline{h(x)}\, h(x) f(x)$$
$$= |h(x)|^2 f(x)$$

$$T_h T_h^* f(x) = T_h T_{\bar{h}} f(x) = h(x)\, T_{\bar{h}} f(x) = h(x)\, \overline{h(x)}\, f(x)$$
$$= |h(x)|^2 f(x)$$

hence $T_h^* T_h f = T_h T_h^* f$, $\forall f \in L^2([0,1])$

$\implies T_h^* T_h = T_h T_h^*$ (T_h is normal)

if $h \in \mathcal{C}_{\mathbb{R}}([0,1])$ then $\overline{h(x)} = h(x)$

and so $T_h^* = T_{\bar{h}} = T_h$ (T_h is self-adjoint)

If $h \in \mathcal{C}_{\mathbb{F}}([0,1])$ such that $|h(x)| = 1$, $\forall x \in [0,1]$

then $T_h^* T_h = T_h T_h^* = I_{L^2} \implies T_h$ is <u>unitary</u>.

3) $T : \mathbb{C}^n \to \mathbb{C}^n$, fix a basis $\{e_1, \dots, e_n\}$, then

$T = M[a_{ij}]_{i,j=1,\dots,n}$. Then $T^* = T \iff$

$a_{ij} = \overline{a_{ji}}$ that is to say the matrix

is self-adjoint: $a_{ij} = \overline{a_{ji}}$.

Theorem. H complex Hilbert space. $T \in B(H)$. Then T is unitary $\Leftrightarrow$ T is an isometry and T is onto (T is an isometric isomorphism).

Lemma. H complex Hilbert space and $\mathcal{U}$ the set of unitary operators in $B(H)$. Then

1) If $u \in \mathcal{U} \Rightarrow u^* \in \mathcal{U}$ and $\|u\|_{B(H)} = \|u^*\|_{B(H)} = 1$

2) If $u_1, u_2 \in \mathcal{U} \Rightarrow u_1 u_2 \in \mathcal{U}$ and $u_1^{-1}, u_2^{-1} \in \mathcal{U}$

3) $\mathcal{U}$ is a closed subset of $B(H)$.

SPECTRUM OF AN OPERATOR

Remark. If $T : \mathbb{C}^n \to \mathbb{C}^n$ linear operator then $\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \underbrace{\dim \mathbb{C}^n}_{= n}$.

Hence: T is injective $\Leftrightarrow$ T is onto.

This is not true if we consider operators between infinite-dimensional Hilber spaces.

Examples. 1) S unilater shift, $\operatorname{Ker} S = \{0\}$ and $\operatorname{Im} S = \overline{\operatorname{Im} S} \subsetneq \ell^2$

2) $T_c : \ell^2 \to \ell^2$, $c = \left(\frac{1}{n}\right)_{n \in \mathbb{N}_+}$

$T_c x = \left(\frac{x_n}{n}\right)$, $\forall x = (x_n) \in \ell^2$

Then $T_c^* = T_{\bar{c}}$ $\quad \bar{c} = (\bar{c}_n)$

here $\bar{c} = c$

hence $T_c^* = T_c$ self-adjoint

T_c injective $\Leftrightarrow$ $\operatorname{Ker} T_c = \{0\}$

$$T_c x = 0 \iff \left(\frac{x_n}{n}\right) = (0) \iff \frac{x_n}{n} = 0, \ \forall n \in \mathbb{N}_+$$
$$\iff x_n = 0$$

Hence $\operatorname{Ker} T_c = \{0\}$

T_c is not onto. In fact, consider the seq.

$$y = \left(\tfrac{1}{n}\right) \underset{\sim c}{\in} \ell^2 \qquad \left(\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \Rightarrow y \in \ell^2\right)$$

$$T_c x = y \iff \frac{x_n}{n} = \frac{1}{n}, \quad \forall n \in \mathbb{N}_+$$
$$\iff x_n = 1, \quad \forall n \in \mathbb{N}_+$$
$$x = (1) \notin \ell^2$$

So $\operatorname{Im} T_c \subset \ell^2$.

$$\overline{\operatorname{Im} T_c} = \left(\operatorname{Ker} T_c^*\right)^{\perp} \overset{T_c^* = T_c}{=} \left(\operatorname{Ker} T_c\right)^{\perp} = \{0\}^{\perp} = \ell^2$$