Spectrum of an operator $T \in \mathcal{B}(H)$

H complex Hilbert space, $T \in \mathcal{B}(H)$, $\lambda \in \mathbb{C}$.

Then $T - \lambda I \in \mathcal{B}(H)$

Definition. The spectrum of T is the set

$$\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible} \}$$

$$\rho(T) = \mathbb{C} \setminus \sigma(T) \quad \text{"the resolvent set of } T\text{"}.$$

$$\mathbb{C} = \sigma(T) \cup \rho(T) \quad \text{and} \quad \sigma(T) \cap \rho(T) = \phi .$$

The following situations may occur

$T - \lambda I$

- case 1: $T - \lambda I$ injective
  - $\operatorname{Im}(T-\lambda I) = H \iff T-\lambda I$ onto $\iff \lambda \in \rho(T)$
  - $\operatorname{Im}(T-\lambda I) \neq H$ but $\overline{\operatorname{Im}(T-\lambda I)} = H \iff \lambda \in \sigma_c(T)$
  - $\operatorname{Im}(T-\lambda I) \subsetneq H$ and it is not dense in $H \iff \lambda \in \sigma_r(T)$
- case 2: $T - \lambda I$ not injective
  $\iff \operatorname{Ker}(T-\lambda I) \neq \{0\}$
  $\iff \exists\, x \in H \setminus \{0\} : (T-\lambda I)x = 0$
  that is, $Tx = \lambda x$
  $\iff \lambda \in \sigma_p(T)$ (λ eigenvalue of T)
  x: eigenvector of T

$\sigma_p(T)$: point spectrum of T (set of eigenvalues)

$\sigma_c(T)$: continuous spectrum of T
$\sigma_r(T)$: residual spectrum of T

Hence $$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$
disjoint union

Property. If $T \in \mathcal{B}(H)$ such that $T^* = T$ then $\sigma_r(T) = \phi$.

Remark. If H is a finite-dimensional complex Hilbert space then for $T \in \mathcal{B}(H)$ we have $\sigma(T) = \sigma_p(T)$.

Example. Consider $T_c \in \mathcal{B}(\ell^2)$, $c = (\frac{1}{n})_{n \in \mathbb{N}_+}$
then recall $T_c^* = T_c$ and T_c is injective and $\operatorname{Im} T_c$ is dense in ℓ^2. So, for $\lambda = 0$ hence $T_c - 0I = T_c$ then $0 \in \sigma_c(T_c)$.
Moreover, since $T_c^* = T_c \Rightarrow \sigma_r(T_c) = \phi$
$\sigma_p(T)$? We have to study: $T_c x = \lambda x$
$\forall x = (x_n) \in \ell^2$ this means we have to solve

$$\frac{x_n}{n} = \lambda x_n, \quad \forall n \in \mathbb{N}_+ \quad \Longleftrightarrow$$

$$\left(\frac{1}{n} - \lambda\right) x_n = 0, \quad \forall n \in \mathbb{N}_+$$

$$(1-\lambda) x_1 = 0 \Longleftrightarrow \lambda = 1, \quad x_1 \in \mathbb{C} \setminus \{0\}, \ x_n = 0, \forall n \geq 2$$

$$\left(\frac{1}{2} - \lambda\right) x_2 = 0 \Longleftrightarrow \lambda = \frac{1}{2}, \quad x_2 \in \mathbb{C} \setminus \{0\}, \ x_n = 0, \forall n \neq 2$$

$$\vdots$$

$$\left(\frac{1}{n} - \lambda\right) x_n = 0 \Longleftrightarrow \lambda = \frac{1}{n}, \quad x_n \in \mathbb{C} \setminus \{0\}, \ x_m = 0, \forall m \neq n$$

So, $\sigma_p(T_c) = \left(\frac{1}{n}\right)_{n\in\mathbb{N}_+}$

Example. $\mathcal{H}$ complex Hilbert space, $T = \mu I$, $\mu \in \mathbb{C}$.

Then $\sigma(T) = \sigma_p(T) = \{\mu\}$.

Solution.
$$\begin{aligned}\sigma(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\} \\ &= \{\lambda \in \mathbb{C} : \mu I - \lambda I \text{ " " "}\} \\ &= \{\lambda \in \mathbb{C} : (\mu-\lambda) I \text{ " " "}\} \\ &= \{\mu\}\end{aligned}$$

for $\lambda = \mu$ $(\mu - \lambda) I = 0$ mapping $\Rightarrow$ any $x \in \mathcal{H}\setminus\{0\}$ is an eigenvector $\Rightarrow$ μ is an eigenvalue

that is, $\sigma(T) = \sigma_p(T)$.

There are bounded operators on infinite-dimensional Hilbert spaces that have no eigenvalues!

Example. The unilateral shift $S \in B(\ell^2)$ does not have eigenvalues.

Solution. By contradiction, assume $\lambda \in \sigma_p(S)$.

That is $\exists\, x = (x_n) \in \ell^2 \setminus \{0\}$ / $Sx = \lambda x$

$\Leftrightarrow$ $(0, x_1, x_2, \ldots, x_n, \ldots) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n, \ldots)$

$$\lambda x_1 = 0$$
$$\lambda x_2 = x_1$$
$$\lambda x_3 = x_2$$
$$\vdots$$
$$\lambda x_n = x_{n-1}, \quad n \geq 2$$
$$\vdots$$

if $\lambda = 0$ $\Rightarrow$ $x_n = 0$, $\forall n \geq 1$ $\Leftrightarrow$ $x = 0$ contradiction!

if $\lambda \neq 0$, we must have $x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow x_3 = 0$
$\Rightarrow x_n = 0, \ \forall n \quad \Rightarrow x = 0$ contradiction!

Hence $\sigma_p(S) = \emptyset$.

Remark. Observe that $0 \in \sigma_r(S)$ S: unilateral shift. $S = S - 0I$ is injective and $\operatorname{Im} S$ is not dense in $\ell^2 \Rightarrow 0 \in \sigma_r(S)$.

Theorem. H complex Hilbert space, $T \in B(H)$. Then

1) If $|\lambda| > \|T\|_{B(H)}$ then $\lambda \notin \sigma(T)$

2) $\rho(T)$ is open (equivalently $\sigma(T)$ is closed).

Proof. 1) $|\lambda| > \|T\|_{B(H)} \iff 1 > |\lambda|^{-1}\|T\|_{B(H)}$

$\iff 1 > \|\lambda^{-1}T\|_{B(H)}$

Hence the operator $I - \lambda^{-1}T$ is invertible by the Neumann series theorem

Observe that $-\lambda I$ is invertible because $\lambda \neq 0$ hence $(-\lambda I)(I - \lambda^{-1}T) = T - \lambda I$ is invertible because product of invertible operators $\Rightarrow$ $\lambda \in \rho(T)$.

2) We need the following lemma.

Lemma. $A, B \in B(H)$, A invertible and B / $\|B\|_{B(H)} < \|A^{-1}\|^{-1}_{B(H)}$. Then $A+B$ is invertible.

Proof of the lemma. $A + B = A(I + A^{-1}B)$

$\|A^{-1}B\|_{B(H)} \leq \|A^{-1}\|_{B(H)} \|B\|_{B(H)} < \|A^{-1}\|_{B(H)} \cdot \|A^{-1}\|^{-1}_{B(H)} = 1$

hence, by the Neumann series theorem, $I + A^{-1}B$ is invertible $\Rightarrow$ $A+B$ is invertible because product of invertible operators. ∎

We shall prove $\rho(T)$ is open : $\forall \lambda_0 \in \rho(T)$ $\exists \delta > 0$: $B_{\lambda_0}(\delta) = \{ \lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta \} \subset \rho(T)$

in other words, if $|\lambda - \lambda_0| < \delta$ then $T - \lambda I$ is invertible.

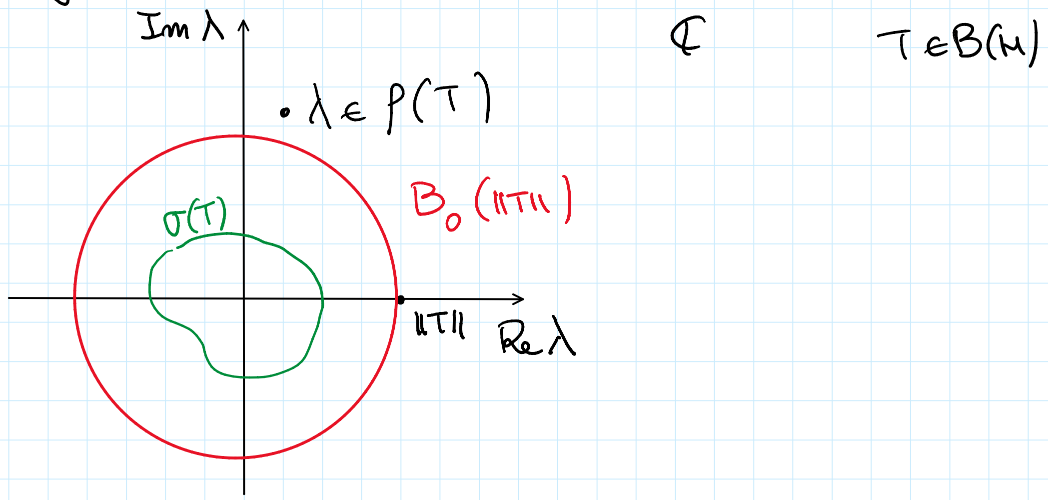
$$T - \lambda I = (T - \lambda_0 I) + (\lambda_0 I - \lambda I) = \underbrace{T - \lambda_0 I}_{\substack{\text{invertible} \\ = A}} + \underbrace{(\lambda_0 - \lambda) I}_{= B}$$

$$\| B \|_{B(H)} < \| A^{-1} \|^{-1}_{B(H)}$$

$$\| (\lambda_0 - \lambda) I \|_{B(H)} = |\lambda_0 - \lambda| \, \overbrace{\| I \|_{B(H)}}^{=1} = |\lambda_0 - \lambda|$$

Hence we choose $\delta := \| A^{-1} \|^{-1}_{B(H)} = \| (T - \lambda_0 I)^{-1} \|^{-1}_{B(H)}$

So the assumptions of the previous lemma are satisfied and $T - \lambda I$ is invertible.



Hence $\sigma(T)$ is closed and bounded $\Leftrightarrow \sigma(T)$ is a compact set in $\mathbb{C}$.

Property. If H complex Hilbert space, $T \in B(H)$ then $\sigma(T) \neq \emptyset$.

Lemma. H complex Hilbert space and $T \in B(H)$. Then $\sigma(T^*) = \{ \bar{\lambda} : \lambda \in \sigma(T) \}$.

Proof. $\lambda \notin \sigma(T) \iff T-\lambda I$ is invertible

$\iff (T-\lambda I)^*$ is invertible

$\iff T^* - \bar{\lambda} I$ is invertible

$\iff \bar{\lambda} \notin \sigma(T^*)$

$\sigma(T^*) = \{ \bar{\lambda} : \lambda \in \sigma(T) \}$.

Property. $S \in B(\ell^2)$ unilateral shift. Then

1) If $\lambda \in \mathbb{C} : |\lambda| < 1$ then $\lambda \in \sigma(S^*)$; in particular λ is an eigenvalue of S^*

2) $\sigma(S) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}$.

Proof. Consider $\lambda \in \mathbb{C} : |\lambda| < 1$, we have to find an eigenvector $x = (x_n) \in \ell^2 \setminus \{0\}$ such that

$$S^* x = \lambda x \iff$$

$$\begin{aligned} x_2 &= \lambda x_1 \\ x_3 &= \lambda x_2 \\ x_4 &= \lambda x_3 \\ &\vdots \\ x_n &= \lambda x_{n-1}, \quad n \geq 2 \\ &\vdots \end{aligned}$$

take $x_1 = 1 \Rightarrow x_2 = \lambda \Rightarrow x_3 = \lambda^2 \dots \Rightarrow x_n = \lambda^{n-1}$

Let us check that the sequence $(x_n = \lambda^{n-1})_{n \in \mathbb{N}_+}$ is in ℓ^2: $\displaystyle\sum_{n=1}^{\infty} |\lambda^{n-1}|^2 \overset{n-1=m}{=} \sum_{m=0}^{\infty} |\lambda^m|^2$

$$= \sum_{m=0}^{\infty} (|\lambda|^2)^m$$

Since $|\lambda| < 1$ by assumption $\Rightarrow |\lambda|^2 < 1$ hence the series above is a geometric series with $q = |\lambda|^2 < 1 \Rightarrow$ it is convergent

So $x = (\lambda^{n-1}) \in \ell^2$ and it is an eigenvector for $S^* x = \lambda x$. Hence $\{\lambda : |\lambda| < 1\} \subseteq \sigma_p(S^*)$

2) $S = (S^*)^*$ so that the set

$$\{\bar{\lambda} : |\lambda| < 1\} \subseteq \sigma(S)$$

since $|\bar{\lambda}| = |\lambda|$ we have that

$$\{\lambda : |\lambda| < 1\} \subseteq \sigma(S)$$

We know: $\|S\|_{B(\ell^2)} = 1$ hence by the theorem above, $\forall \lambda : |\lambda| > 1 \Rightarrow \lambda \notin \sigma(S)$

Since $\sigma(S)$ is closed we get that

$$\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

Remark. One can prove:

$$\sigma_p(S^*) = B_0(1), \quad \sigma_c(S^*) = \partial B_0(1) := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$

$$\sigma_r(S^*) = \phi$$

$$\sigma_p(S) = \phi, \quad \sigma_c(S) = \partial B_0(1), \quad \sigma_r(S) = B_0(1)$$

Theorem. H complex Hilbert space, $T \in B(H)$.

1) If p is a polynomial, then $\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$

2) If T is invertible then $\sigma(T^{-1}) = \{\lambda^{-1}, \lambda \in \sigma(T)\}$

Notation: $p(\sigma(T)) := \{p(\lambda) : \lambda \in \sigma(T)\}$

Lemma. H complex Hilber space, $U \in B(H)$ unitary operator. Then $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \partial B_0(1)$