Definition. H complex Hilbert space, $T \in B(H)$.

1) The **spectral radius of T** is
$$r_\sigma(T) = \sup\{|\lambda|,\ \lambda \in \sigma(T)\}$$

2) The **numerical range of T**
$$V(T) = \{(Tx, x)_H : \|x\|_H = 1\}$$

Theorem. H complex Hilbert space, $T \in B(H)$ such that $T^* = T$. Then, defining for short $\|T\| = \|T\|_{B(H)}$,

1) $V(T) \subseteq \mathbb{R}$

2) $\sigma(T) \subseteq \mathbb{R}$

3) At least one of $\|T\|$ or $-\|T\|$ is in $\sigma(T)$.

4) $r_\sigma(T) = \sup\{|\tau| : \tau \in V(T)\} = \|T\|$

5) $\forall \mu \in V(T)$, we have
$$\inf\{\lambda : \lambda \in \sigma(T)\} \le \mu \le \sup\{\lambda : \lambda \in \sigma(T)\}$$

Proof of 1) — 4)

1) We know $T^* = T$, hence
$$(Tx, x) \overset{(A)}{=} (x, \overset{=T}{T^*}x) = (x, Tx) = \overline{(Tx, x)}$$
$$(Tx, x) = \overline{(Tx, x)} \iff (Tx, x) \in \mathbb{R},$$
$\forall x \in H$ so in particular for $x : \|x\| = 1$
$$\Rightarrow \quad V(T) \subseteq \mathbb{R}$$

2) We use the following lemma:

Lemma. $T \in B(H)$, T normal $(T^*T = TT^*)$. Then $\sigma(T) \subseteq \overline{V(T)}$.

Observe that T self-adjoint $\Rightarrow$ T is normal so by the previous lemma $\sigma(T) \subseteq \overline{V(T)} \subseteq \overline{\mathbb{R}} = \mathbb{R}$

$\Rightarrow \sigma(T) \subseteq \mathbb{R}$.

3) The result is trivial if $\|T\| = 0 \iff T = 0$ (0-mapping) hence $0 \in \sigma(T)$ and

$0 = \|T\| \in \sigma(T)$

Now assume $T \in B(H)$, $T^* = T$ with $\|T\| = 1$

Since $\|T\| = \sup_{\|x\|_H = 1} \|Tx\|_H$, by def. of supremum

there exists a sequence $\{x_n\} \subset H$ with $\|x_n\|_H = 1$ such that $\|Tx_n\|_H \to \|T\| = 1$ as $n \to \infty$

$$\|(I - T^2)x_n\|^2 = ((I-T^2)x_n, (I-T^2)x_n))$$
$$= \|x_n\|^2 + \|T^2 x_n\|^2 - 2\,\mathrm{Re}(T^2 x_n, x_n)$$

$$(T^2 x_n, x_n) \overset{T^* = T}{\underset{(A)}{=}} (Tx_n, Tx_n) = \|Tx_n\|^2 \geq 0$$

$\Rightarrow \mathrm{Re}(T^2 x_n, x_n) = \|Tx_n\|^2$

$$\|T^2 x_n\| \leq \|T\| \|Tx_n\| \leq \underbrace{\|T\|^2}_{=1} \|x_n\| = \|x_n\|$$

$$\|(I - T^2)x_n\|^2 \leq \underbrace{\|x_n\|^2}_{=1} + \underbrace{\|x_n\|^2}_{=1} - 2\|Tx_n\|^2$$

$$= 2 - 2\underbrace{\|Tx_n\|^2}_{\downarrow\, n\to\infty \atop 1} \to 0, \text{ as } n \to \infty$$

By the comparison theorem, $\|(I - T^2)x_n\|^2 \to 0$ hence $\|(I-T^2)x_n\| \to 0$ as $n \to \infty$.

This implies that the operator $I - T^2$ is not invertible by the characterization for invertible operators: in this case we cannot find $\alpha > 0$ such that $\|(I-T^2)x\| \geq \alpha \|x\|$ $\forall x \in X$. This is not case since we have found a sequence $\{x_n\}$ / $\|x_n\| = 1$ such that $\|(I-T^2)x_n\| \to 0$

$$I - T^2 = (I-T)(I+T)$$

hence at least one between $I-T$ and $I+T$ is not invertible hence either $\lambda = +1$ or $\lambda = -1$ belongs to $\sigma(T)$. Recall $\|T\|=1$

If $T \in B(H) \setminus \{0\}$, $T^* = T$, we can write $T = \|T\| \cdot \frac{T}{\|T\|} = \|T\| R$ with $R \in B(H)$, $R^* = R$ and $\|R\| = 1$ hence

$\sigma(T) = \{\|T\| \lambda : \lambda \in \sigma(R)\}$ hence either $\|T\|$ or $-\|T\|$ is in $\sigma(T)$.

a) By definition $r_\sigma(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ from item 3) we know that either $\|T\|$ or $-\|T\|$ is in $\sigma(T)$ $\Rightarrow$

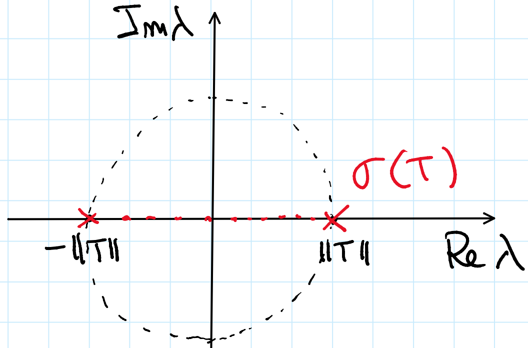
$$\begin{aligned}\|T\| &\leq \sup\{|\lambda| : \lambda \in \sigma(T)\} \\ &\leq \sup\{|\lambda| : \lambda \in V(T)\} \\ &= \sup\{|(Tx, x)| : \|x\| = 1\}\end{aligned}$$

$$|(Tx, x)| \leq \|Tx\| \, \|x\| \leq \|T\| \underbrace{\|x\|^2}_{=1} = \|T\|$$

hence $\|T\| \leq \sup\{|\lambda| : \lambda \in \sigma(T)\} = r_\sigma(T)$

$$\begin{aligned}&\leq \sup\{|(Tx,x)| : \|x\| = 1\} \\ &\leq \|T\|\end{aligned}$$

hence $\quad \|T\| = r_\sigma(T) = \sup\{|(Tx,x)| : \|x\|=1\}$



$\mathbb{C} \qquad T\in B(H),\ T^*=T$

One between $\|T\|$ and $-\|T\|$ is in $\sigma(T)$

	Operators	Spectrum (subset of)
1)	$T\in B(H)$	$\mathbb{C}$
2)	T self-adjoint	$\mathbb{R}$
3)	T unitary	$\{\lambda : \lvert\lambda\rvert=1\}$

Example. H complex Hilbert space, $T\in B(H)$, $T^*=T$.

Then

1) T^n is self-adjoint $\forall n\in\mathbb{N}_+$

2) $\|T^n\|_{B(H)} = \|T\|^n_{B(H)}$

Solution. 1) $\quad (T^n)^* = (\underbrace{T\cdot T\cdots T}_{n\text{-times}})^* = \underbrace{T^*\cdot T^*\cdots T^*}_{n\text{-times}}$

$$\overset{T^*=T}{=} \underbrace{T\cdot T\cdots T}_{n\text{-times}} = T^n$$

T^n is self-adjoint.

2) By Item 4) of the previous theorem,

$\|T^n\|_{B(H)} = r_\sigma(T^n) = \sup\{|\lambda| : \lambda\in\sigma(T^n)\}$

$$= \sup\{ |\lambda^n| : \lambda \in \sigma(T) \}$$
$$= \sup\{ |\lambda|^n : \lambda \in \sigma(T) \}$$

$y = x^n$, $x \geq 0$ $\Rightarrow$ it is continuous and increasing on $[0, +\infty)$

$x = |\lambda| \geq 0$

hence $\|T^n\|_{B(H)} = \left(\sup\{ |\lambda| : \lambda \in \sigma(T) \} \right)^n$
$$= (r_\sigma(T))^n$$
$$= \|T\|^n_{B(H)} .$$

Example. A is a $n \times n$ self-adjoint matrix $A^* = A$ with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$. Then
$$\|A\| = \max\{ |\lambda_1|, \dots, |\lambda_n| \} .$$

Solution. By Item a) of the previous theorem
$$\|A\| = r_\sigma(A) = \sup\{ |\lambda| : \lambda \in \sigma(A) \}$$
$$= \sup\{ |\lambda| : \lambda \in \sigma_p(A) \}$$
$$= \max\{ |\lambda_1|, \dots, |\lambda_n| \} .$$

Property. H complex Hilbert space, $T \in B(H)$, $T^* = T$. Then t. f. c. a. e. :

1) $\sigma(T) \subseteq [0, +\infty)$
2) $(Tx, x) \geq 0$, $\forall x \in H$
3) $(Tx, x) \geq 0$, $\forall x \in H : \|x\|_H = 1$.

Definition. H complex Hilbert space, $T \in B(H)$, $T^* = T$. T is said *positive* if
$$(Tx, x) \geq 0, \quad \forall x \in H \qquad (\Leftrightarrow \sigma(T) \subseteq [0, +\infty))$$

Examples. H complex Hilbert space.

1) $O, I \in B(H)$ and self-adjoint are positive operators

2) If $T \in B(H)$ then T^*T and TT^* are positive

3) $R, S \in B(H)$ positive operators, $\alpha > 0$ then
$R+S$ is positive and αR is positive.

Solution. 1) $(Ox, x) = 0 \quad \forall x$

$(Ix, x) = \|x\|^2 \geq 0, \quad \forall x$

2) $(T^*T)^* = T^*\underbrace{(T^*)^*}_{=T} = T^*T$ self-adjoint

Similarly, TT^* is self-adjoint

$(T^*Tx, x) \overset{(A)}{=} (Tx, \underbrace{(T^*)^*}_{=T}x) = (Tx, Tx) = \|Tx\|^2 \geq 0$

T^*T is positive. Similarly, TT^* is positive.

3) $((R+S)x, x) = \underbrace{(Rx, x)}_{\geq 0} + \underbrace{(Sx, x)}_{\geq 0} \geq 0$

$((\alpha R)x, x) = \underbrace{\alpha}_{>0} \underbrace{(Rx, x)}_{\geq 0} \geq 0$

Definition. $T \in B(H)$, H complex Hilbert space.
A **square root** of T is an operator $R \in B(H)$: $R^2 = T$.

Theorem. H complex Hilbert space, $T \in B(H)$, $T^* = T$ and T positive. Then

1) There exists a positive operator $R \in B(H)$ such that $R^2 = T$ (R square root of T)

2) The square root R is unique: if $A \in B(H)$

positive : $A^2 = T$ then $A = R$.

Property. $T \in \mathcal{B}(H)$, $T^* = T$. Then eigenvectors corresponding to different eigenvalues are orthogonal. (Extension of the case of symmetric matrices).

Proof. Consider $\lambda_1, \lambda_2 \in \sigma_p(T)$: $\lambda_1 \neq \lambda_2$ then there exist $x_1, x_2 \neq 0$ such that $Tx_1 = \lambda_1 x_1$ and $Tx_2 = \lambda_2 x_2$.

$$(Tx_1, x_2) = (\lambda_1 x_1, x_2) = \lambda_1 (x_1, x_2)$$

$$\overset{(A)}{=} (x_1, \underbrace{(T^*)^*}_{=T} x_2) = (x_1, Tx_2) = (x_1, \lambda_2 x_2) = \overset{\in\mathbb{R}}{\lambda_2} (x_1, x_2)$$

hence $\lambda_1 (x_1, x_2) = \lambda_2 (x_1, x_2) \iff$

$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} (x_1, x_2) = 0$$

$\Rightarrow (x_1, x_2) = 0$ so x_1 and x_2 are orthogonal. ∎

Definition. An *orthogonal projection on H* (H complex Hilbert space) is an operator $P \in \mathcal{B}(H)$ such that $P = P^* = P^2$

(the orthogonal projection is a projection that is self-adjoint.)

An orthogonal projection P is a positive operator:

$$(Px, x) \overset{P=P^2}{=} (P^2 x, x) \overset{(A)}{=} (Px, \underset{P^*}{P}x) = \|Px\|^2 \geq 0,$$

$\forall x \in H$.

Example. $P : \mathbb{C}^2 \to \mathbb{C}^2 \; : \; P(x,y) = (x,0)$

then P is an orthogonal projection.

Solution. P is trivially linear $\Rightarrow P \in B(\mathbb{C}^2)$

$P^2 = P$ obvious

Let us check $P^* = P$, $\forall (u,w) \in \mathbb{C}^2$,

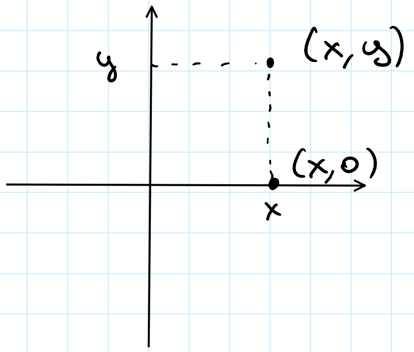
$$(P(x,y), (u,w)) = ((x,0), (u,w)) = x\bar{u}$$

$$((x,y), P(u,w)) = ((x,y), (u,0)) = x\bar{u}$$

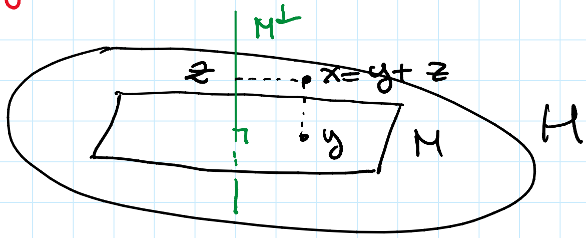
$\forall (x,y), (u,w) \in \mathbb{C}^2 \Rightarrow P^* = P$.

P projects vectors vertically downwards or orthogonally onto $\operatorname{Im} P = \{ (x,0) : x \in \mathbb{C} \} \cong \mathbb{C}$

$\mathbb{C}^2$



Theorem. H complex Hilbert space, $M \subset H$ closed subspace of H. Then there exists an orthogonal projection $P_M \in B(H)$: $\operatorname{Im} P_M = M$ $\operatorname{Ker} P_M = M^\perp$ and $\|P_M\|_{B(H)} \le 1$. P_M is called the **orthogonal projection of H onto M**.



$P_M x = y$

Vice versa:

Theorem. P orthogonal projection, then $\operatorname{Im} P$ is a closed subspace of H and $P = P_{\operatorname{Im} P}$.

Lemma. $M \subset H$, M closed subspace of H. If P_M is the orthogonal projection of H onto M then $I - P_M$ is the orthogonal projection of H onto $M^{\perp}$.

Corollary. $M \subset H$ closed subspace of H, $\{e_n\}_{n=1}^{J}$ o.n.b. for M (where J positive integer or ∞) P_M orthogonal projection of H onto M. Then

$$P_M x = \sum_{n=1}^{J} (x, e_n) e_n .$$