COMPACT OPERATORS

Definition. X, Y normed spaces over the same scalar field $\mathbb{F}$.

1) $T \in L(X, Y)$ is compact if $\forall \{x_n\} \subset X$ bounded sequence $\Rightarrow$ $\{Tx_n\} \subset Y$ admits a convergent subsequence: $\exists \{Tx_{n_k}\} \subseteq \{Tx_n\}$ | $Tx_{n_k} \to y \in Y$ as $k \to \infty$.

2) $T \in L(X, Y)$ is compact if $\forall A \subseteq X$ bounded set $\Rightarrow$ $T(A) \subseteq Y$ is relatively compact ($\Longleftrightarrow$ $\overline{T(A)}$ is a compact set in Y).

Let us check that these 2 definitions are equivalent:

1) $\Rightarrow$ 2) Consider $A \subseteq X$ bounded set; we want to show that $T(A)$ is relatively compact. Hence consider any sequence $\{y_n\} \subseteq T(A)$ $\Rightarrow$ $\exists \{x_n\} \subseteq A$ | $Tx_n = y_n$. Since A is bounded $\Rightarrow$ $\{x_n\}$ is bounded so by Def. 1) $\exists \{Tx_{n_k}\} \subseteq \{Tx_n\}$ | $Tx_{n_k} \to y \in Y$, that is, $T(A)$ is rel. compact.

2) $\Rightarrow$ 1) take any $\{x_n\} \subseteq X$ bounded, call $A = \{x_n\}$, then $T(A)$ is rel. comp. $T(A) = \{Tx_n\}$ $\Rightarrow$ $\exists \{Tx_{n_k}\} \subseteq \{Tx_n\}$ | $Tx_{n_k} \to y \in Y$ as $k \to \infty$.

Definition. Given $T \in L(X, Y)$, we define the "rank of T", denoted by $r(T)$,

$$r(T) := \dim \operatorname{Im} T$$

If $r(T) < \infty$ we say that T is a finite rank operator.

Proposition. If $T \in B(X,Y)$ and of finite rank then T is compact.

Proof. Consider $\{x_n\} \subseteq X$ bounded : $\exists M > 0$.

$\|x_n\|_X \leq M \quad \forall n$

$\|Tx_n\|_Y \overset{T \text{ bounded}}{\leq} \|T\|_{B(X,Y)} \|x_n\| \leq \|T\|_{B(X,Y)} M, \quad \forall n$

Hence $\{Tx_n\} \subset \operatorname{Im} T$ is a bounded seq.

Since T is a finite rank operator $\Rightarrow \operatorname{Im} T$ is finite dimensional hence by Bolzano-Weierstrass Theorem, the bounded sequence $\{Tx_n\}$ admits a convergent subsequence : $Tx_{n_k} \to y \in Y$.

Bolzano-Weierstrass Theorem

X finite-dimensional normed space. Then any bounded sequence $\{x_n\}$ admits a convergent subsequence $\{x_{n_k}\}$ | $x_{n_k} \to x \in X$.

Definition. The set of compact operators between X and Y is denoted by $K(X,Y)$.

Hence $K(X,Y) \subseteq L(X,Y)$.

Theorem. X, Y normed spaces and $T \in K(X,Y)$ then $T \in B(X,Y)$. Hence $K(X,Y) \subseteq B(X,Y)$.

Proof. By contradiction.

$T \in B(X,Y) \iff \exists C > 0 : \|Tx\|_Y \leq C, \quad \forall x \in X : \|x\|_X = 1$

We negate the previous statement:

$\forall C > 0 \quad \exists x \in X : \|x\|_X = 1 \,/\, \|Tx\|_Y > C.$

Choose $C = n$, $\forall n \in \mathbb{N}_+ \quad \exists x_n : \|x_n\|_X = 1$ such that $\|Tx_n\|_Y > n$

Since T is compact and $\{x_n\}$ is bounded ($\|x_n\|_X = 1, \forall n$) then $\{Tx_n\}$ admits a subsequence $\{Tx_{n_k}\} \,/\, Tx_{n_k} \to y \in Y$

Any convergent sequence is bounded: $\exists M > 0$:

$\|Tx_{n_k}\|_Y \le M, \quad \forall k$

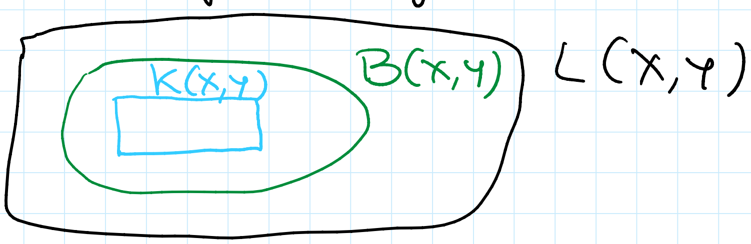
This is a contradiction since by construction

$\|Tx_{n_k}\|_Y > n_k \longrightarrow \infty$ as $k \to \infty$. ∎

Theorem. X, Y, Z normed spaces, $T \in \mathcal{B}(X,Y)$, $S \in \mathcal{B}(Y,Z)$. Then

1) If at least one between T and S is compact then ST is compact ($ST \in \mathcal{K}(X,Z)$).

2) If $T_1, T_2 \in \mathcal{K}(X,Y)$ then $\forall \alpha, \beta \in \mathbb{F}$, $\alpha T_1 + \beta T_2 \in \mathcal{K}(X,Y)$. Hence $\mathcal{K}(X,Y)$ is a subspace of $\mathcal{B}(X,Y)$.



Example. X, Y normed spaces. If either X or Y is is finite dimensional and $T \in \mathcal{B}(X,Y)$ then $T \in \mathcal{K}(X,Y)$.

Solution. First case: assume X finite dimensional

$$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim X < \infty$$

$$\Rightarrow r(T) = \dim \operatorname{Im} T < \infty$$

So T is a finite rank operator, from what we have shown above T is compact.

Second case: Y is finite dimensional. Then

$$\operatorname{Im} T \subseteq Y \quad \Rightarrow \quad r(T) = \dim \operatorname{Im} T < \infty$$

hence T is a finite rank operator $\Rightarrow$ T is compact.

Theorem. X infinite-dimensional normed space. Then the identity $I \in B(X)$ is not compact.

Proof. Consider $A = B_0(1) = \{x \in X : \|x\|_X < 1\}$ then $I(B_0(1)) = B_0(1)$ that is not relatively compact since $\overline{B_0(1)} = \{x \in X : \|x\|_X \leq 1\}$ is not compact since X is infinite dimensional.

Corollary. X, Y normed space with X infinite dim. and $T \in K(X,Y)$ then T is not invertible.

Proof. By contradiction, $\exists\, T^{-1} \in B(Y, X)$ such that

$$\underbrace{T^{-1}}_{T^{-1} \text{ is bounded}} \; \underbrace{T}_{T \text{ is compact}} = I \quad \text{over } X$$

then I (the product) is compact. This contradicts the previous theorem.

Property. X, Y normed spaces. If $T \in K(X,Y)$ then $\operatorname{Im} T$ and $\overline{\operatorname{Im} T}$ are separable.

Sketch of proof. $X = \bigcup_{n \geq 0} B_0(n)$

$$\operatorname{Im} T = T(X) = \bigcup_{n \geq 0} \underbrace{T(B_0(n))}_{\text{relatively compact set}}$$

Recall: (M, d) metric space, A (relatively) compact set, $A \subseteq M$ then A is separable.

So $\operatorname{Im} T$ is countably union of separable sets $\Rightarrow$ $\operatorname{Im} T$ is separable.

Hence $\overline{\operatorname{Im} T}$ is separable since $\operatorname{Im} T \subseteq \overline{\operatorname{Im} T}$ and any $A \subseteq \operatorname{Im} T$ dense in $\operatorname{Im} T$ is also dense $\overline{\operatorname{Im} T}$.

Remark. Even though X is "big" (not separable) if $T \in K(X,Y)$, $\operatorname{Im} T$ is "small" (separable).

Property. X, Y normed spaces, $T \in L(X,Y)$. Then T is compact $\Leftrightarrow$ $T(B_0(1))$ is relatively compact in Y ($\Leftrightarrow$ $\overline{T(B_0(1))}$ is compact in Y).

Proof. "$\Rightarrow$" it follows by definition of T compact

"$\Leftarrow$" $T(B_0(1))$ is relatively compact

$\Leftrightarrow$ $T(B_0(r))$ is rel. compact, $\forall r > 0$

$\forall \; x \in B_0(r) \Rightarrow x = r\frac{x}{r}$; setting $y = \frac{x}{r}$

we have $\left\| \frac{x}{r} \right\|_X = \frac{1}{r} \|x\|_X < \frac{1}{r} \cdot r = 1 \Rightarrow$

$y \in B_0(1)$. Vice versa: $\forall y \in B_0(1)$
$x = ry \in B_0(r)$
hence $B_0(r) = \{ ry : \|y\|_X < 1 \} = r\{y : \|y\|_X < 1\}$
$= r B_0(1)$

$$T(B_0(r)) = T(r(B_0(1))) \overset{T \text{ lin.}}{=} rT(B_0(1))$$

Let us show T is compact: $\forall A \subseteq X$ bounded
$\Rightarrow \exists r > 0 : A \subseteq B_0(r)$ hence

$$T(A) \subseteq \underbrace{T(B_0(r))}_{\text{is relatively compact}}$$

$\forall \{y_n\} \subset T(A)$ bounded since
$\{y_n\} \subset T(B_0(r)) \Rightarrow \exists \{y_{n_k}\} \subseteq \{y_n\} /$
$y_{n_k} \to y \in Y$ since $T(B_0(r))$ is rel. comp.

Theorem. X normed space, Y Banach space, $\{T_n\} \subset K(X,Y)$ / $T_n \to T \in B(X,Y)$ then $T \in K(X,Y)$. So $K(X,Y)$ is a closed subspace of $B(X,Y)$.

Remark: Y is a Banach space $\Rightarrow B(X,Y)$ is a Banach space $\Rightarrow K(X,Y)$ is a Banach space since it is a closed subspace of a Banach space.

Corollary. X normed space, Y Banach space, $\{T_n\} \subset B(X,Y)$ sequence of finite rank operators such that $T_n \to T \in B(X,Y)$. Then $T \in K(X,Y)$.

Proof. $\{T_n\} \subset B(X,Y)$ and of finite rank $\Rightarrow$
$\{T_n\} \subset K(X,Y)$.

Exercise. Consider $T_c : \ell^2 \to \ell^2$ with $c = \left(\frac{1}{n}\right)_{n\geq 1}$
$T_c x = \left(\frac{x_n}{n}\right), \quad \forall \; x = (x_n) \in \ell^2.$
Then $T_c \in K(\ell^2)$.

Solution. We know $T_c \in B(\ell^2)$.
The idea is: "construct a sequence $\{T_n\}$ of bounded finite rank operators such that $T_n \to T$ in $B(\ell^2)$.

$T_1 x := (x_1, 0, 0, \dots \;)$
$T_2 x := (x_1, \frac{x_2}{2}, 0, \dots 0 \dots \;)$
$\vdots$
$T_n x := \left(x_1, \frac{x_2}{2}, \dots, \frac{x_n}{n}, 0, 0, \dots \;\right)$
$T_n = T_{c_n}$, where $c_n = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0 \dots\right) \in c_{00} \subset \ell^\infty$

Hence $T_n = T_{c_n} \in B(\ell^2)$. $\to$ finite dimensional
$\operatorname{Im} T_n \subseteq \operatorname{Sp}\{\delta_1, \dots, \delta_n\}$ $\quad \delta_n = (\delta_{nm})_{m \in \mathbb{N}_+}$
$\Rightarrow \dim \operatorname{Im} T_n < \infty$ $\quad \delta_{nm} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$
Hence $\{T_n\} \subset B(\ell^2)$ and it is a sequence of finite rank operators
Let us show $T_n \to T_c$ in $B(\ell^2)$

$$\|T_n x - T_c x\|_{\ell^2}^2 = \left\|\left(0, 0, \dots, 0, -\frac{x_{n+1}}{n+1}, -\frac{x_{n+2}}{n+2}, \dots\right)\right\|_{\ell^2}^2$$
$$= \sum_{k=n+1}^{\infty} \left|-\frac{x_k}{k}\right|^2$$

$$= \sum_{k=m+1}^{\infty} \left| \frac{x_k}{k^2} \right|^2 \leq \frac{1}{(m+1)^2} \sum_{k=m+1}^{\infty} |x_k|^2$$

$$\leq \frac{1}{(m+1)^2} \|x\|_{\ell^2}^2$$

So, $\|(T_m - T_c)x\|_{\ell^2}^2 \leq \frac{1}{(m+1)^2} \|x\|_{\ell^2}^2$

$$\|(T_m - T_c)x\|_{\ell^2} \leq \frac{1}{m+1} \|x\|_{\ell^2}, \quad \forall x \in \ell^2$$

Hence $\|T_m - T_c\|_{B(\ell^2)} \leq \frac{1}{m+1} \to 0$, as $m \to \infty$

by the comparison theorem $\|T_m - T_c\|_{B(\ell^2)} \to 0$, $m \to \infty$ that is, $T_m \to T_c$ so that by the previous result T_c is compact.

Theorem E1. X normed space, H Hilbert space. $T \in K(X, H)$. Then there exists $\{T_n\} \subset B(X,H)$ of finite rank operators $|$ $T_n \to T$ in $B(X,H)$.

Lemma. H Hilbert space, $T \in B(H)$, then $r(T) = r(T^*)$. In particular, T is of finite rank $\iff$ T^* is of finite rank.

Theorem. H complex Hilbert space and $T \in B(H)$. Then T is compact $\iff$ T^* is compact.

Proof. $\Rightarrow$ assume $T \in K(H)$ then by Theorem E1 there exists a sequence $\{T_n\} \subset B(H)$ of finite rank operators such that $T_n \to T$. By the previous lemma $T_n^* \in B(H)$ is of finite rank

$$\|T_n^* - T^*\|_{B(H)} = \|(T_n - T)^*\|_{B(H)} = \|T_n - T\|_{B(H)}$$

By assumption, $T_n \to T$ hence $T_n^* \to T^*$
so $T^* \in K(H)$ because limit of a sequence of compact operators.

$\Leftarrow$ $T^* \in K(H)$, write $T = (T^*)^*$ and apply the previous step. ☒

Theorem (Ascoli - Arzelà Theorem)

$\mathcal{F} \subset C_{\mathbb{F}}([a,b])$ is relatively compact $\Leftrightarrow$

1) $\mathcal{F}$ is bounded $\Leftrightarrow$ $\exists M > 0 : \|f\|_\infty \le M$, $\forall f \in \mathcal{F}$.

2) $\mathcal{F}$ is equicontinuous $\Leftrightarrow$ $\forall \varepsilon > 0\ \exists \delta > 0 :$
$\forall f \in \mathcal{F}$ $\forall x_1, x_2 \in [a,b]$, $|x_1 - x_2| < \delta$
$|f(x_1) - f(x_2)| < \varepsilon$

(this conditions means that any $f \in \mathcal{F}$ is uniformly continuous and "δ" is the same for any $f \in \mathcal{F}$).

Remark. Ascoli - Arzelà Thrm implies:

If $\{f_n\} \subset C_{\mathbb{F}}([a,b])$ then if

1) $\{f_n\}$ is bounded
2) $\{f_n\}$ is equicontinuous

$\Rightarrow$ $\exists \{f_{n_k}\} \subseteq \{f_n\}$ such that
$f_{n_k} \to f \in C_{\mathbb{F}}([a,b])$, as $k \to \infty$.