Example. $\mathcal{F} = \{ f \in \mathcal{C}^1([a,b]) : |f'(x)| \leq 1, \ \forall x \in [a,b] \}$
$\subset \mathcal{C}([a,b])$

$\mathcal{F}$ is not relatively compact in $\mathcal{C}([a,b])$.

Solution. By Lagrange Theorem, $\forall x_1, x_2 \in [a,b] \ \exists c \in (x_1, x_2) /$

$$|f(x_1) - f(x_2)| = |f'(c)(x_1 - x_2)| = \underbrace{|f'(c)|}_{\leq 1} \cdot |x_1 - x_2| \leq |x_1 - x_2|$$

$\forall \varepsilon > 0$ choose $\delta = \varepsilon$ so that $\forall f \in \mathcal{F}, \forall x_1, x_2$:
$|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$

$\implies \mathcal{F}$ is equicontinuous

$\mathcal{F}$ is not bounded. Consider the functions
$f_n(x) = n, \quad \forall x \in [a,b], \quad n \in \mathbb{N}_+$
$f_n \in \mathcal{C}^1([a,b])$ and $f_n'(x) = 0, \ \forall x \in [a,b]$
$\implies f_n \in \mathcal{F}, \quad \forall n \in \mathbb{N}_+$

$$\|f_n\|_\infty = \sup_{x \in [a,b]} |f_n(x)| = n \to \infty \text{ as } n \to \infty$$

so $\{f_n\} \subset \mathcal{F}$ is not bounded $\implies \mathcal{F}$ is not bounded. Hence by Ascoli-Arzelà theorem $\mathcal{F}$ is not relatively compact.

Example. $\mathcal{F} = \{ f \in \mathcal{C}^1([a,b]) \mid |f'(x)| \leq 1, |f(x)| \leq 1, \ \forall x \in [a,b] \}$

Then $\mathcal{F}$ is equicontinuous.

Solution. We use Ascoli-Arzelà theorem and we show:

1) $\mathcal{F}$ is equicontinuous (as in the previous example).

2) $\mathcal{Y}$ is bounded

$\forall f \in \mathcal{Y}$, $\|f\|_\infty = \sup_{x\in[a,b]} \underbrace{|f(x)|}^{\leq 1} \leq 1$.

Example. $T : \mathcal{C}([a,b]) \to \mathcal{C}([a,b])$,

$Tf(x) = \int_a^b k(x,y)\, f(y)\, dy$, integral operator with kernel $k \in \mathcal{C}([a,b]\times[a,b])$. Then T is compact.

Solution. We already proved $T \in \mathcal{B}(\mathcal{C}([a,b]))$ and $\|T\| \leq \|k\|_\infty (b-a)$.

In order to prove that T is compact it is enough to show that $T(B_0(1))$ is relatively compact. $T(B_0(1)) \subset \mathcal{C}([a,b])$

We use Ascoli-Arzelà theorem:

1) $T(B_0(1))$ is bounded:

$\forall f \in B_0(1)$,

$$\|Tf\|_\infty \leq \|T\|_{\mathcal{B}(\mathcal{C}([a,b]))} \overbrace{\|f\|_\infty}^{<1} < \|T\|_{\mathcal{B}(\mathcal{C}([a,b]))}$$

2) $T(B_0(1))$ is equicontinuous.

$\forall f \in B_0(1)$, $\forall x_1, x_2 \in [a,b]$

$$|Tf(x_2) - Tf(x_1)| = \left| \int_a^b k(x_2,y) f(y)\, dy - \int_a^b k(x_1,y) f(y)\, dy \right|$$

$$\leq \int_a^b |k(x_2,y) - k(x_1,y)| \cdot \overbrace{|f(y)|}^{<1}\, dy$$

$$< \int_a^b |k(x_2,y) - k(x_1,y)|\, dy$$

Since $k \in \mathcal{C}([a,b]\times[a,b])$ then by Heine-Cantor k is uniformly continuous on $[a,b]\times[a,b]$ so that, $\forall \varepsilon>0\ \exists \delta>0 : \forall x_1, x_2, y \in [a,b]$ |

$$|x_1 - x_2| < \delta \iff \|(x_1,y)-(x_2,y)\|_2 < \delta$$

$$\|(x_1,y)-(x_2,y)\|_2 = \sqrt{(x_1-x_2)^2+(y-y)^2} = |x_1-x_2|$$

$$\Rightarrow |k(x_2,y) - k(x_1,y)| < \frac{\varepsilon}{b-a}$$

Hence, $|Tf(x_2) - Tf(x_1)| < \int_a^b \frac{\varepsilon}{b-a}\, dy$

$$= \frac{\varepsilon}{\cancel{b-a}}\,(\cancel{b-a}) = \varepsilon$$

$\forall f \in B_0(1)$.

$\Rightarrow$ (A-A) $T(B_0(1))$ is rel. compact.

Exercise. $k \in L^2([a,b]\times[a,b])$ and consider the integral operator $Tf(x) = \int_a^b k(x,y)\, f(y)\, dy$ with $f \in L^2([a,b])$. Then $T \in \mathcal{K}(L^2([a,b]))$.

Solution. We know that $T \in \mathcal{B}(L^2([a,b]))$ and $\|T\|_{\mathcal{B}(L^2([a,b]))} \leq \|k\|_{L^2([a,b]\times[a,b])}$.

Consider $\{e_n\}_{n\in\mathbb{N}_+}$ an o.n.b. for $L^2([a,b])$.

$(e_n \otimes e_m)(x,y) := e_n(x)\, e_m(y)$ tensor product

$\{e_n \otimes e_m\}_{n,m\in\mathbb{N}_+}$ is an o.n.b. for $L^2([a,b]\times[a,b])$

$$k(x,y) = \sum_{n,m\in\mathbb{N}_+} c_{nm}\,(e_n\otimes e_m)(x,y)$$

$$c_{nm} = (k, e_n\otimes e_m)$$

with uncondit. convergence in $L^2([a,b]\times[a,b])$

Define $k_N(x,y) = \sum_{n,m=1}^{N} c_{nm}(e_n \otimes e_m)(x,y)$

$\in L^2([a,b]\times[a,b])$

We have $k_N \to k$ in $L^2([a,b]\times[a,b])$

Define $T_N f(x) = \int_a^b k_N(x,y) f(y)\, dy$

Since $k_N \in L^2([a,b]\times[a,b]) \Rightarrow T_N \in B(L^2([a,b]))$

$$T_N f(x) = \int_a^b \sum_{n=1}^{N} \sum_{m=1}^{N} c_{nm}\, e_n(x) e_m(y) f(y)\, dy$$

$$= \sum_{n=1}^{N} e_n(x) \sum_{m=1}^{N} c_{nm} \underbrace{\int_a^b e_m(y) f(y)\, dy}_{= c \in \mathbb{C}}$$

$\in Sp\{e_1, \dots, e_N\}$

$\dim \operatorname{Im} T_N < \infty$

$\Rightarrow T_N$ is a finite rank operator

By construction, $\|k - k_N\|_{L^2([a,b]\times[a,b])} \to 0, \; N \to \infty$

$T - T_N$ is the integral operator with kernel $k - k_N$ so $\|T - T_N\|_{B(L^2([a,b]))} \le \|k - k_N\|_{L^2} \xrightarrow{N\to\infty} 0$

By the comparison theorem $T_N \to T$ in $B(L^2([a,b]))$ hence T is compact.

Definition. H infinite-dimensional complex Hilbert space, $T \in B(H)$. If for an o.n.b. $\{e_n\} \subset H$ we have $\sum_{n=1}^{\infty} \|Te_n\|_H^2 < \infty$ then T is called **Hilbert-Schmidt operator**.

Theorem. H complex, infinite-dimensional Hilbert space, $T \in B(H)$. Consider $\{e_n\}$, $\{f_m\}$ two o.n.b. for H. Then

1) $$\sum_{n=1}^{\infty} \|Te_n\|_H^2 = \sum_{m=1}^{\infty} \|T^* f_m\|_H^2 = \sum_{m=1}^{\infty} \|Tf_m\|_H^2$$

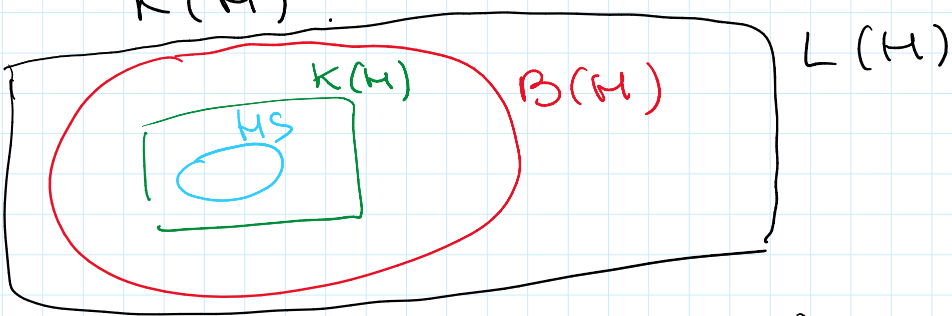
where the values of the sums may be either finite or ∞.

(The Hilbert-Schmidt property does not depend on the o.n.b.)

2) T is Hilbert-Schmidt (HS) $\iff$ T^* is HS

3) If T is HS $\Rightarrow$ T is compact

4) The set of HS operators is a subspace of $K(H)$.



Example. $k \in L^2([a,b] \times [a,b])$ then

$$Tf(x) = \int_a^b k(x,y) f(y)\, dy$$ is a HS operator.

Solution. $\{e_n\}$ o.n.b. for $L^2([a,b])$

$$\sum_{n=1}^{\infty} \| T e_n \|^2_{L^2([a,b])} = \sum_{n=1}^{\infty} \int_a^b |T e_n(x)|^2 \, dx$$

$$= \sum_{n=1}^{\infty} \int_a^b \underbrace{\left| \int_a^b k(x,y) e_n(y) \, dy \right|^2}_{\geq 0} dx$$

Tonelli theorem

$$= \int_a^b \sum_{n=1}^{\infty} \left| \int_a^b k(x,y) e_n(y) \, dy \right|^2 dx$$

$$= \int_a^b \sum_{n=1}^{\infty} | (k(x,\cdot), \bar{e}_n) |^2 \, dx$$

$\{\bar{e}_n\}$ is an o.n.b. since $\{e_n\}$ is an o.n.b.

Parseval's Theorem

$$= \int_a^b \| k(x,\cdot) \|^2_{L^2([a,b])} \, dx$$

$$= \int_a^b \int_a^b |k(x,y)|^2 \, dy \, dx$$

$$= \| k \|^2_{L^2([a,b] \times [a,b])} < \infty$$

T is HS

Let us check $\{\bar{e}_n\}$ is an o.n.b.

$\forall F \in L^2([a,b])$ $\quad \| F \|^2_{L^2([a,b])} = \| \bar{F} \|^2_{L^2([a,b])}$

Parseval's T.

$$= \sum_{n=1}^{\infty} | (\bar{F}, e_n) |^2$$

$$= \sum_{n=1}^{\infty} | (F, \bar{e}_n) |^2$$

the characterization for o.n.b. gives $\{\bar{e}_n\}$ is an o.n.b. ∎

SPECTRAL THEORY FOR COMPACT OPERATORS ON COMPLEX HILBERT SPACES

If H is a finite-dimensional Hilbert space then $\sigma(T) = \sigma_p(T)$ $\quad \forall T \in K(H)$

From now on, H is an infinite-dimensional Hilbert space and $T \in K(H)$.

Theorem 1. $0 \in \sigma(T)$.

Proof. By contradiction, if $0 \in \rho(T)$ then $T = T - 0I$ is invertible. This is a contradiction since H is infinite dimensional.

Theorem 2. If $\lambda \in \sigma(T) \setminus \{0\}$ then $Ker(T-\lambda I)$ is finite dimensional.

Proof. By contradiction, assume $Ker(T-\lambda I)$ is infinite dimensional. Since $T-\lambda I \in B(H) \Rightarrow$ $Ker(T-\lambda I) \subseteq H$ is a closed subspace of H. Hence $Ker(T-\lambda I)$ is a Hilbert space.

Consider $\{e_n\} \subset Ker(T-\lambda I)$ an o.n. sequence for $Ker(T-\lambda I)$. Hence $(T-\lambda I)e_n = 0 \Leftrightarrow$ $Te_n = \lambda e_n$. Since $\|e_n\|_H = 1$, $\forall n$, $\{e_n\}$ is a bounded sequence $\Rightarrow$ since T is compact $\{Te_n\}$ admits a convergent subsequence $Te_{n_k} \to y \in H$. But this is not possible because any convergent seq. is a Cauchy seq.

$$\|Te_n - Te_m\|_H^2 = \|\lambda e_n - \lambda e_m\|_H^2$$
$$= (\lambda e_n - \lambda e_m, \lambda e_n - \lambda e_m)$$
$$= 2|\lambda|^2 > 0$$

So the Cauchy property does not hold and we get the contradiction. ☒

Theorem 3. If $\lambda \in \sigma(T) \setminus \{0\} \Rightarrow \operatorname{Im}(T - \lambda I)$ is closed.

Corollary. If $\lambda \in \sigma(T) \setminus \{0\}$ then

$$\operatorname{Im}(T - \lambda I) = \operatorname{Ker}(T^* - \bar{\lambda} I)^{\perp}$$
$$\operatorname{Im}(T^* - \bar{\lambda} I) = \operatorname{Ker}(T - \lambda I)^{\perp}.$$

Theorem 4. If $\lambda \in \sigma(T) \setminus \{0\}$ then $\lambda \in \sigma_p(T)$.

Hence $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.

Remark. It is possible for a compact operator T to have $\sigma_p(T) = \phi$. In this case, $\sigma(T) = \{0\}$.

Example. Consider the operator $T : \ell^2 \to \ell^2$ defined by $Tx = \left(0, \frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, \dots\right)$, $\forall\, x = (x_n) \in \ell^2$.

Show that $T \in K(\ell^2)$, $\sigma_p(T) = \phi$ so $\sigma(T) = \{0\}$.

Solution. • T is linear (check as exercise)

• T is bounded:

$$\|Tx\|_{\ell^2}^2 = \sum_{n=1}^{\infty} \left|\frac{x_n}{n}\right|^2 = \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{\le 1} |x_n|^2 \le \sum_{n=1}^{\infty} |x_n|^2 = \|x\|_{\ell^2}^2$$

$T \in B(\ell^2)$ and $\|T\|_{B(\ell^2)} \le 1$.

$$T_n x := \left(0, \frac{x_1}{1}, \dots, \frac{x_n}{n}, 0, 0, 0, \dots\right)$$

$T_n \in B(\ell^2)$, $\forall n \in \mathbb{N}_+$

$T_n x \in \mathrm{Sp}\{\delta_1, \delta_2, \dots, \delta_{n+1}\}$

$\Rightarrow \dim \mathrm{Im}\, T_n < \infty \Rightarrow T_n$ is a finite rank operator

$T_n \to T$ in $B(\ell^2)$ (check as exercise).

$\sigma_p(T) = \phi$ by contradiction, assume $\exists\, \lambda \in \sigma_p(T)$ and a $x \in \ell^2$ / $x \neq 0$ and $Tx = \lambda x$

$$0 = \lambda x_1$$
$$\frac{x_1}{1} = \lambda x_2$$
$$\frac{x_2}{2} = \lambda x_3$$
$$\vdots$$
$$\frac{x_n}{n} = \lambda x_{n+1}$$
$$\vdots$$

If $\lambda = 0 \Rightarrow x_1 = 0, x_2 = 0, \dots, x_n = 0 \Rightarrow x = 0$ contradiction!

If $x_1 = 0$ $(\lambda \neq 0) \Rightarrow x_2 = 0 \Rightarrow x_3 = 0 \dots$
$\Rightarrow x = 0$ contradiction!

Hence $\sigma_p(T) = \phi$ Since $0 \in \sigma(T) \Rightarrow$
$\sigma(T) = \{0\}$.

Theorem 5. The set $\sigma_p(T)$ (possibly empty) is at most countably infinite. If $\{\lambda_n\}$ is any sequence of distinct eigenvalues of T then $\lim_{n\to\infty} \lambda_n = 0$.

Example. Find the spectrum of $T: L^2([0,1]) \to L^2([0,1])$ defined by $Tf(x) = \int_0^1 e^{x-y} f(y)\,dy$

Solution. $e^{x-y} \in \mathcal{C}([0,1]\times[0,1]) \subset L^2([0,1]\times[0,1])$

$T \in K(L^2([0,1]))$. Hence $0 \in \sigma(T)$.

$0 \in \sigma_p(T)$? Study $Tf(x) = 0$ for $f \neq 0$, $f \in L^2([0,1])$. That is,

$$\underbrace{e^x}_{\neq 0!} \int_0^1 e^{-y} f(y)\,dy = 0 \iff \int_0^1 e^{-y} f(y)\,dy = 0 \quad (I)$$

$$f(y) = \begin{cases} a, & 0 \le x \le \frac{1}{2} \\ b, & \frac{1}{2} < x \le 1 \end{cases}$$

find $a, b \in \mathbb{C}$ / (I) is satisfied:

$$(I) \iff \int_0^{\frac{1}{2}} e^{-y} a\,dy + \int_{1/2}^1 e^{-y} b\,dy = 0$$

$$\iff a\,[-e^{-y}]_0^{1/2} + b\,[-e^{-y}]_{1/2}^1 = 0$$

$$a(1 - e^{-1/2}) + b(e^{-1/2} - e^{-1}) = 0$$

Choose for instance $b = 1 \implies$

$$a = \frac{e^{-1} - e^{-1/2}}{1 - e^{-1/2}}$$

$\implies 0 \in \sigma_p(T)$.

Consider now $\lambda \in \sigma_p(T)$: $\lambda \neq 0$ we have to study $Tf(x) = \lambda f(x) \iff$

$$e^x \underbrace{\int_0^1 e^{-y} f(y)\,dy}_{= \alpha \in \mathbb{C}\setminus\{0\}} = \lambda f(x) \quad (II)$$

$$f(x) = \underbrace{\frac{\alpha}{\lambda}}_{= \beta \in \mathbb{C}\setminus\{0\}} e^x = \beta e^x$$

Insert f in (II):

$$\cancel{e^x} \int_0^1 e^{-y} \cancel{\beta} e^y \, dy = \lambda \cancel{\beta} \cancel{e^x}$$

$$\iff \int_0^1 dy = \lambda \iff \lambda = 1$$

hence $\sigma(T) = \sigma_p(T) = \{0, 1\}$.