SPECTRAL THEORY FOR COMPACT SELF-ADJOINT OPERATORS

Recall: If H is a finite-dimensional complex Hilbert space, $\dim H = n$ and $T : H \to H$ linear operator $| \; T^* = T$ then there exists an o.n.b. $\{e_j\}_{j=1}^n$ of eigenvectors of T for H. So $\forall x \in H$, $x = \sum_{j=1}^n (x, e_j) e_j$

$$Tx = \sum_{j=1}^n (x, e_j) \underbrace{Te_j}_{= \lambda_j e_j} = \sum_{j=1}^n \lambda_j (x, e_j) e_j$$

where λ_j is the eigenvalue associated with e_j.

Consider now a complex Hilbert space H.

Theorem. If $T \in K(H)$ such that $T^* = T$ then at least one between $\|T\|_{B(H)}$, $-\|T\|_{B(H)}$ is in $\sigma_p(T)$.

Proof. If $T = 0$ (0 mapping) then $\|T\|_{B(H)} = 0$ and $0 \in \sigma_p(T)$ (every $x \in H$, $x \neq 0$ is an eigenvector). If $T \neq 0$ since $T^* = T$ we know that at least one between $\overset{\neq 0}{\|T\|_{B(H)}}$, $-\overset{\neq 0}{\|T\|_{B(H)}}$ is in $\sigma(T) \setminus \{0\} \overset{T \text{ compact}}{=} \sigma_p(T) \setminus \{0\}$.

Theorem. $T \in K(H)$, $T^* = T$, $T \neq 0$. Then the set of non-zero eigenvalues is a collection of real numbers, either finite or countably infinite (sequence). If it is a sequence then it tends to zero. Each non-zero eigenvalue λ_n has finite multiplicity ($\Leftrightarrow \dim \operatorname{Ker}(T - \lambda_n I) < \infty$).

Eigenvectors corresponding to different eigenvalues are orthogonal.

Proof. This is a consequence of the spectral theory for compact operators and self-adjoint operators.

Theorem. The number of non-zero eigenvalues of T (as above) repeated according to multiplicity is equal to $r(T) = \dim \operatorname{Im} T$.

We can construct an o.n.b. $\{e_n\}_{n=1}^{r(T)}$ of eigenvectors for $\overline{\operatorname{Im} T}$ and the operator T has the representation:

$$Tx = \sum_{n=1}^{r(T)} \lambda_n (x, e_n) e_n$$

where $(\lambda_n)_{n=1}^{r(T)}$ is the set of eigenvalues of T.

Corollary. Under the assumptions above, if the $\operatorname{Ker} T = \{0\}$ then the set of eigenvectors $\{e_n\}_{n=1}^{r(T)}$ is an o.n.b. for H.

In particular, if H is infinite-dimensional then T has infinitely many eigenvalues.

Proof. We know by assumption $T^* = T$ so $\operatorname{Ker} T^* = \operatorname{Ker} T = \{0\}$. By the relation

$$\overline{\operatorname{Im} T} = (\operatorname{Ker} T^*)^\perp = (\operatorname{Ker} T)^\perp = \{0\}^\perp = H$$

$\Rightarrow$ H is separable. ■

If H is separable and $\operatorname{Ker} T \neq \{0\}$ then:

Corollary. Under the assumptions above, there exists an o.n.b. of eigenvectors of T for H. This basis has the form:

$$\{e_n\}_{n=1}^{r(T)} \cup \{z_m\}_{m=1}^{m(T)}$$

where $m(T) = \dim \operatorname{Ker} T$, where $\{e_n\}_{n=1}^{r(T)}$ is an o.n.b. for $\overline{\operatorname{Im} T}$ and $\{z_m\}_{m=1}^{m(T)}$ is an o.n.b. for $\operatorname{Ker} T$.

Proof. By the Orthogonal Decomposition Theorem

$$H = \overline{\operatorname{Im} T} \oplus \operatorname{Ker} \underset{T^*}{\overset{\shortparallel}{T}}$$

$\operatorname{Ker} T \subset H$ is closed $\Rightarrow$ $\operatorname{Ker} T$ is a Hilbert space separable since H is separable.

Recall: (M, d) metric space. $A, B \subseteq M$, with B separable and $A \subseteq B$. Then A is separable.

Since $\operatorname{Ker} T$ is a separable Hilbert space it admits an o.n.b. $\{z_m\}_{m=1}^{m(T)}$ (eigenvectors associated with the eigenvalue 0). ∎

APPLICATIONS

Remark. Consider a $m \times m$ matrix $A : \mathbb{C}^n \to \mathbb{C}^n$.

Consider the following equations:

(a) $Ax = y$

(b) $Ax = 0$

Then either (a) has a unique solution $\forall y \in \mathbb{C}^n$ or (b) has a solution $x \neq 0$. That is,

(i) A is bijective or

(ii) A is not bijective $\Leftrightarrow \operatorname{Ker} A \neq \{0\}$

$m := \dim \operatorname{Ker} A > 0$

Since $\operatorname{Ker} A^* = (\operatorname{Im} A)^{\perp}$ and we have the orthog. decomposition:

$$\mathbb{C}^n = \underset{\substack{\uparrow \\ \text{is closed}}}{\operatorname{Im} A} \oplus \operatorname{Ker} A^*$$

Since $\dim \mathbb{C}^n = n \Rightarrow n = \dim \operatorname{Im} A + \dim \operatorname{Ker} A^*$

Recall: $\dim \operatorname{Ker} A + \dim \operatorname{Im} A = n$

$$\Rightarrow \quad \dim \operatorname{Ker} A^* = \dim \operatorname{Ker} A = m$$

Hence the equation in (a) has solution $\Leftrightarrow$ $y \in \operatorname{Im} A \Leftrightarrow y \perp \operatorname{Ker} A^*$. Consider a basis $\{v_1, \dots, v_m\}$ for $\operatorname{Ker} A^*$ then y must satisfy m orthogonal relations:

$$\begin{cases} (y, v_1) = 0 \\ \vdots \\ (y, v_m) = 0 \end{cases}$$

Fredholm integral equations

(1) First Kind Fredholm int. eq.

$$\int_a^b K(x,y) f(y)\, dy = g(x)$$

where $k(x,y)$ is known, $g(x)$ is known, $f(y)$ is unknown.

(2) Second kind Fredholm int. eq.

$$f(x) - \mu \int_a^b k(x,y) f(y)\,dy = g(x), \qquad \mu \in \mathbb{C} \setminus \{0\}$$

k, g known, f unknown.

If we define $Tf(x) = \int_a^b k(x,y) f(y)\,dy$ the integral operator with kernel k, then previous equations can be rewritten as:

(1) $Tf = g$

(2) $If - \mu Tf = g \iff (I - \mu T) f = g$

$$\iff -\mu\left(-\frac{1}{\mu} I + T\right) f = g$$

If we call $\lambda := \frac{1}{\mu}$ then (2) becomes:

$$(T - \lambda I) f = -\lambda g$$

$\tilde{g} := -\lambda g$ then (2) is equivalent to:

$$(T - \lambda I) f = \tilde{g} \qquad (\lambda \neq 0)$$

Definition. X, Y normed spaces over $\mathbb{F}$. Given $T \in \mathcal{B}(X,Y)$. Consider the problem

$$Tu = v \qquad (\#)$$

The problem is said **well-posed** if the following conditions occur:

1) $\forall v \in Y \; \exists u \in X$ solution of (#)

2) The solution u is unique
3) The solution u depends continuously on v.

Remark. If we work with $X = Y = L^2([a,b])$ and consider $g \in L^2([a,b])$ and $k \in L^2([a,b]\times[a,b])$ then $T \in K(L^2([a,b]))$ hence T is not invertible hence the problem (1) is not well-posed.

Theorem (Fredholm Alternative)

H complex infinite-dimensional Hilbert space. Consider the equations:

1) $(T - \lambda I)x = 0$ $\qquad (T^* - \bar{\lambda} I)y = 0$

2) $(T - \lambda I)x = p$ $\qquad (T^* - \bar{\lambda} I)x = q$

for $T \in K(H)$ and $\lambda \neq 0$. Then one or the other of the following alternative holds:

a) Each of the homogeneous equations in 1) has only the solution $x = 0$ (resp. $y = 0$), while the corresponding inhomogeneous equations in 2) have unique solutions x (resp. y) for any p (resp. q) in H.

b) There exists a finite number $m_\lambda \in \mathbb{N}_+$ such that each of the equations in 1) has exactly m_λ linearly independent solutions x_n, $n = 1, \dots, m_\lambda$

(resp. y_n, $n = 1, \dots, m_\lambda$), while the corresponding inhomog. equations in 2) have solutions $\iff$ $p, q \in H$ satisfy:

$$(p, y_n) = 0 \qquad \text{resp.} \quad (q, x_n) = 0, \quad \forall n = 1, \dots, m_\lambda$$

Sketch of proof. $T \in K(H)$ and $\lambda \neq 0$. Then we have only two possibilities: either $\lambda \in \rho(T)$ ($\Rightarrow \bar\lambda \in \rho(T^*)$) then $T - \lambda I$ (resp. $T^* - \bar\lambda I$) is invertible hence we get the alternative a). Otherwise $\lambda \in \sigma_p(T)$ ($\Rightarrow \bar\lambda \in \sigma_p(T^*)$)

$$\dim \operatorname{Ker}(T - \lambda I) = \dim \operatorname{Ker}(T^* - \bar\lambda I) = m_\lambda < \infty$$

$$\operatorname{Im}(T - \lambda I) = \operatorname{Ker}(T^* - \bar\lambda I)^\perp$$

$$\operatorname{Im}(T^* - \bar\lambda I) = \operatorname{Ker}(T - \lambda I)^\perp$$

since T is compact (T^* comp.) $\lambda \neq 0$ ($\bar\lambda \neq 0$) eigenvalue of T (resp. T^*) $\Rightarrow$ $\operatorname{Im}(T - \lambda I)$ (resp. $\operatorname{Im}(T^* - \bar\lambda I)$) is closed.

Corollary. Consider $T \in K(H)$ and $\lambda \neq 0$. If $(T - \lambda I)x = 0$ has only the solution $x = 0$ then $T - \lambda I$ is invertible and the equation

$$(T - \lambda I)x = p$$

has the unique solution $x = (T - \lambda I)^{-1} p$ which depends continuously on p. Hence the problem is well-posed.

Corollary. For $T \in K(H)$, define $S = I + T$. Then,

1) $\dim \operatorname{Ker} S = \dim \operatorname{Ker} S^* < \infty$
2) $\operatorname{Im} S$ is closed.

Remark. Observe that $S \in B(H)$ since sum of bounded operators but $S \notin K(H)$ if H is infinite dimensional. In fact, $I = S - T$ if by contradiction S were compact then
$I = S - T \in K(H)$ since $K(H)$ is a subspace and this a contradiction.

Corollary. With the notations above, either

(1) $Sx = y$ has a unique solution x for every $y \in H$ ($\Leftrightarrow$ S is invertible)

or

(2) $\operatorname{Ker} S$ is not trivial and $Sx = y$ has at least one solution $\Leftrightarrow$ $y \in \operatorname{Ker} S^{*\perp}$

Given a basis $\{z_1, \dots, z_m\}$ for $\operatorname{Ker} S^*$ this means that y must satisfy the orthogonality relations:

$$(y, z_j) = 0, \quad \forall j = 1, \dots, m.$$

Example. Study the equation

$$f(x) + \int_a^b k(x,y) f(y)\, dy = g(x) \qquad (I)$$

for $k \in L^2([a,b] \times [a,b])$ and $g \in L^2([a,b])$.

Solution. We set $Tf(x) := \int_a^b k(x,y) f(y) dy$ so that the equation in (I) becomes:

$$(I+T)f = g.$$

Hence by what we have seen above, either (I) is uniquely solvable ($\Leftrightarrow$ $I+T$ is invertible) or, in order to have solutions, g must satisfy a certain number of orthogonality relations.

Example. Consider $k(x,y) = -e^{x-y}$ on $[0,1]$ and $g(x) = 0$, $\forall x \in [0,1]$

T^* is the integral operator with kernel

$$\tilde{k}(x,y) = \overline{k(y,x)} = -e^{y-x}$$

$S = I+T$ hence $S^* = I+T^*$

$\operatorname{Ker} S^*$?

$$S^* u(x) = 0 \Leftrightarrow$$

$$u(x) + \int_0^1 -e^{y-x} u(y) dy = 0$$

$$\Leftrightarrow \quad u(x) = e^{-x} \underbrace{\int_0^1 e^y u(y) dy}_{\beta \in \mathbb{C}} = \beta e^{-x}$$

inserting $u(x) = \beta e^{-x}$ in the equation

$$\beta e^{-x} + \int_0^1 -\cancel{e^y} \cdot e^{-x} \beta \cancel{e^{-y}} dy = 0$$

$$\Leftrightarrow \quad \beta e^{-x} - \beta e^{-x} \underbrace{\int_0^1 dy}_{=1} = 0$$

$\forall \beta \in \mathbb{C}$.

$$\operatorname{Ker} S^* = \{\beta e^{-x},\ \beta \in \mathbb{C}\} = \operatorname{Sp}\{e^{-x}\} \neq 0$$

Since $\dim \operatorname{Ker} S = \dim \operatorname{Ker} S^* = 1$, then

$S = I + T$ is not invertible and in order to have solution to

$$(I + T) f = g$$

g must satisfy the orthogonality relation

$$(g, e^{-x}) = 0 \iff \int_0^1 g(x) e^{-x} \, dx = 0$$