Lecture 1a

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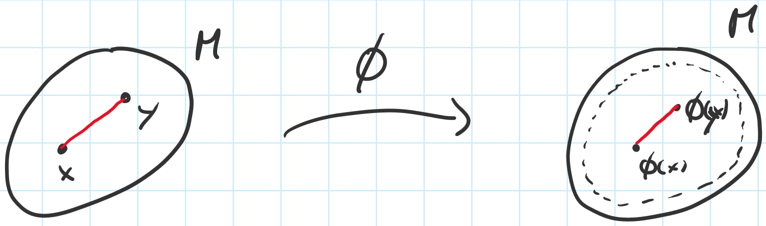
Banach fixed point theorem

(M,d) metric space. $\phi : M \to M$ is called strict contraction if

$$\exists\, 0 \le L < 1 \quad \forall x,y \in M : \; d(\phi(x),\phi(y)) \le L\, d(x,y)$$



ϕ is continuous!

$z \in M$ is a fixed point of ϕ if $\phi(z) = z$.

Theorem If (M,d) is complete and ϕ strict contraction, then ϕ has a unique fixed point z. If

$$x_0 \in M, \quad x_{n+1} = \phi(x_n) \quad (n \in \mathbb{N})$$

$$[\, x_0, \phi(x_0) = x_1, \; x_2 = \phi(x_1) = \phi(\phi(x_0)), \dots \,]$$

then

$$x_n \xrightarrow{n \to +\infty} z$$

Moreover:

a) a-priori estimate: $d(x_n, z) \le \dfrac{L^n}{1-L}\, d(x_1, x_0)$

b) a-posteriori-estimate: $d(x_n, z) \le \dfrac{L}{1-L}\, d(x_n, x_{n-1})$.

PROOF:

Uniqueness Let z, z' be fixed points.

$$d(z,z') = d(\phi(z),\phi(z')) \le L\, d(z,z')$$

$$\Rightarrow \quad 0 \le \underbrace{(1-L)}_{>0}\, d(z,z') \le 0 \quad \Rightarrow d(z,z') = 0$$

$$\Rightarrow z = z'$$

$$\underbrace{\qquad}_{>0}$$

$$\Rightarrow z = z'$$

<u>Existence</u>

$$\begin{aligned} d(x_{k+1}, x_k) &= d(\phi(x_k), \phi(x_{k-1})) \\ &\le L\, d(x_k, x_{k-1}) \\ &\le \dots \le L^k d(x_1, x_0) \end{aligned}$$

If $m > n$

$$d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_n)$$

$$\left[\ d(x,y) \le d(x,z) + d(z,y) \right]$$

$$\le \sum_{e=n}^{m-1} L^e d(x_1, x_0)$$

$$\le d(x_1, x_0)\, L^n \underbrace{\sum_{e=0}^{+\infty} L^e}_{= \frac{1}{1-L}}$$

$$L^N \xrightarrow{N \to +\infty} 0$$

$$\Rightarrow d(x_m, x_n) \le \frac{L^N}{1-L} d(x_1, x_0) \qquad \forall\, m > n \ge N$$

$\Rightarrow (x_n)_n$ is Cauchy sequence

M complete $\Rightarrow$ $\exists\, z \in M : \quad x_n \xrightarrow{n \to +\infty} z$

Then

$$z \longleftarrow x_{n+1} = \phi(x_n) \longrightarrow \phi(z)$$

$$\Rightarrow z = \phi(z).$$

$$d(x_n, z) \xleftarrow{m \to +\infty} d(x_n, x_m) \le \frac{L^n}{1-L} d(x_1, x_0) \qquad (m > n)$$

$\Rightarrow$ a)

$$d(x_m, x_n) \le \sum_{\ell=0}^{m-n-1} d(x_{n+\ell+1}, x_{n+\ell})$$

$$\le \sum_{\ell=0}^{m-n-1} L^{\ell+1} d(x_n, x_{n+1})$$

Take limit $m \longrightarrow +\infty \Rightarrow$

$$d(z, x_n) \le \underbrace{\sum_{\ell=0}^{+\infty} L^{\ell+1}}_{= \frac{L}{1-L}} d(x_n, x_{n+1})$$

$\blacksquare$

Application: Initial value problems

$f : [a,b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous with

$$\| f(t,x) - f(t,x') \| \le C \, \|x - x'\| \quad \forall t \in [a,b] \; \forall x, x' \in \mathbb{R}^n$$

with some constant $C \geq 0$. ("Lipschitz continuity")

Picard-Lindelöf Theorem Let $x_0 \in \mathbb{R}^n$, $t_0 \in [a,b]$.

$\Rightarrow$ Exists unique $x \in C^1([a,b], \mathbb{R}^n)$ of

$$\left.\begin{aligned} x'(t) &= f(t, x(t)), \quad a \le t \le b \\ x(t_0) &= x_0 . \end{aligned}\right\} (*)$$

PROOF (Sketch) x is a C^1 solution of $(*)$ iff x is continuous and

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \quad , a \le t \le b . \qquad (**)$$

(fundamental theorem of calculus).

Let $M := C([a,b], \mathbb{R}^n)$ with

Let $M := C([a,b], \mathbb{R}^n)$ with

$$\tilde{d}(f,g) = \max_{a \le t \le b} e^{-(C+1)|t-t_0|} \|g(t) - h(t)\|$$

$$0 < e^{-(C+1)|b-a|} \le e^{-(C+1)|t-t_0|} \le 1$$

$\Rightarrow$ d and $\tilde{d}$ are equivalent

Then $\phi : M \to M$ defined by

$$[\phi(g)](t) = x_0 + \int_{t_0}^{t} f(s, g(s))\, ds \qquad (a \le t \le b)$$

is a strict contraction in $(M, \tilde{d})$ with $L = \frac{C}{1+C} < 1$.

BFP Theorem $\Rightarrow$ ϕ has a unique fixed point

This fixed point is unique solution of $(**)$ hence of $(*)$. $\blacksquare$

<u>Example</u> $x'(t) = x(t)$ $[f(t,x) = x]$

$x(0) = 1$

$x_0 = 1$

$$x_1(t) = [\phi(x_0)](t) = 1 + \int_0^t 1\, ds = 1 + t$$

$$x_2(t) = [\phi(x_1)](t) = 1 + \int_0^t 1 + s\, ds = 1 + t + \frac{t^2}{2}$$

$\vdots$

$$x_n(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots + \frac{t^n}{n!}$$

$$= \sum_{k=0}^{n} \frac{1}{k!} t^k \xrightarrow{n \to +\infty} \sum_{k=0}^{+\infty} \frac{1}{k!} t^k = e^t$$