Lecture 1b

25 November 2021 10:13

Distribution theory

[X Banach space, X' dual space

$x' \in X' \Leftrightarrow x' : X \to \mathbb{C}$ continuous

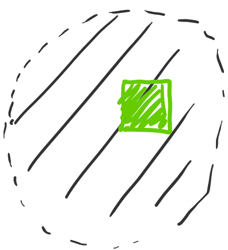
$\Leftrightarrow$ whenever $x_n \to x$ then $x'(x_n) \to x'(x)$

$\Leftrightarrow$ x' is bounded, i.e

$\exists M \geq 0 \ \forall x \in X : |x'(x)| \leq M \|x\|$]

Notation $\Omega \subseteq \mathbb{R}^n$ open set

$K \subset\subset \Omega :\Leftrightarrow K \subset \Omega$ and K compact



1.1 Test functions $\mathcal{D}(\Omega)$

Let $f : \mathbb{R}^n \to \mathbb{C}$ be continuous. Then

closure in $\mathbb{R}^n$

$$\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}} \quad \text{(support of } f\text{)}$$

Example • $f(x) = \max(1-x^2, 0)$

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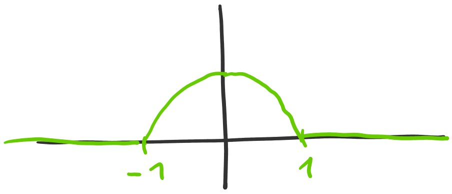
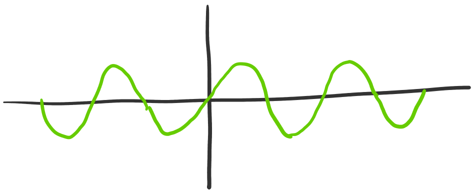


$\operatorname{supp} f = [-1,1]$

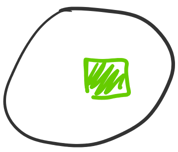
• $f(x) = \sin x$



$\{x \mid \sin x \neq 0\} = \mathbb{R} \setminus \{\pi k \mid k \in \mathbb{Z}\}$

$\operatorname{supp} f = \mathbb{R}$

Definition $\mathcal{D}(\Omega) = \{\phi \in C^\infty(\mathbb{R}^n) \mid \operatorname{supp} \phi \subset\subset \Omega\}$



$\phi : \mathbb{R}^n \to \mathbb{C}$

Is a vector space : $\phi_1, \phi_2 \in \mathcal{D}(\Omega), \lambda \in \mathbb{C} \Rightarrow$

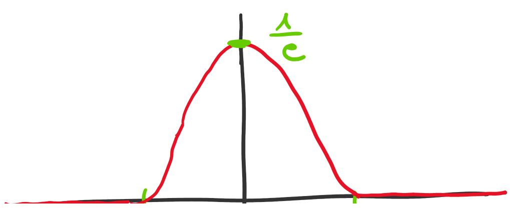
$\phi_1 + \phi_2 \in \mathcal{D}(\Omega)$, $\lambda \phi_1 \in \mathcal{D}(\Omega)$

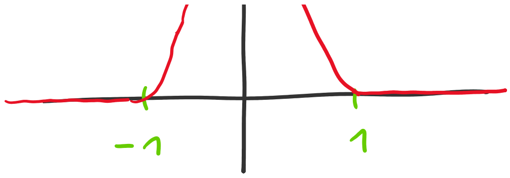
$\uparrow$

$[\operatorname{supp}(\phi_1 + \phi_2) \subseteq \operatorname{supp} \phi_1 \cup \operatorname{supp} \phi_2]$

Example:

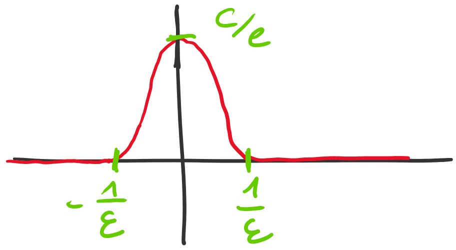
$$\phi(x) = \begin{cases} \exp\left(\frac{1}{\|x\|^2 - 1}\right) & , \|x\| < 1 \\ 0 & , \|x\| \geq 1 \end{cases} \qquad (x \in \mathbb{R}^n)$$





$\phi \in \mathcal{D}(\mathbb{R}^n)$, $\operatorname{supp} \phi = \{ x \mid \|x\| \leq 1 \}$ unit ball in $\mathbb{R}^n$

- $\psi(x) = C\,\phi(\varepsilon x)$ $\quad (\varepsilon > 0,\ C \in \mathbb{C})$



$\psi \in \mathcal{D}(\mathbb{R}^n)$.

"Multi-index notation for derivatives":

$u : \Omega \subseteq \mathbb{R}^n \to \mathbb{C}$, $u(x_1, \dots, x_n)$

$\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}_0$

$$\partial_x^\alpha u = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} u \qquad |\alpha| = \alpha_1 + \dots + \alpha_n$$

$\big[$ $\sin(xy^2)$, $\partial^{(1,3)} \sin(xy^2) = \partial_x \partial_y^3 \sin(xy^2) = \dots$ $\big]$

$u \in C^\infty \Rightarrow$ the order of the derivatives does not matter!

(Schwarz theorem)

Definition Define

$$\|\phi\|_j := \max_{\substack{x \in \mathbb{R}^n, \\ |\alpha| \leq j}} |\partial_x^\alpha \phi(x)| \quad , \ j = 0, 1, 2, \dots$$

Definition $(\phi_k)_k \subset \mathcal{D}(\Omega)$ is said to converge to $\phi \in \mathcal{D}(\Omega)$ if

i) there exists $K \subset\subset \Omega$ s.t. $\operatorname{supp} \phi_k, \operatorname{supp} \phi \subseteq K$

ii) $\|\phi_k - \phi\|_j \longrightarrow 0 \quad \forall j = 0, 1, 2, \dots$

Lemma If $\phi_k \to \phi$ then $\partial_x^\alpha \phi_k \longrightarrow \partial^\alpha \phi$

1.2 Distributions $\mathcal{D}'(\Omega)$

Def $T \in \mathcal{D}'(\Omega)$ if $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ is linear and continuous, i.e.

$$\underbrace{\phi_k \to \phi}_{\text{in } \mathcal{D}(\Omega)} \quad \text{implies} \quad \underbrace{T(\phi_k) \longrightarrow T(\phi)}_{\text{in } \mathbb{C}}$$

Theorem Let $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$ be linear. Then the following are equiv:

a) $T \in \mathcal{D}'(\Omega)$

b) $\forall K \subset\subset \Omega \ \exists C = C(K) \geq 0 \ \exists j = j(K) \in \mathbb{N}$:

$$|T(\phi)| \leq C \|\phi\|_j \quad \forall \begin{array}{c} \phi \in \mathcal{D}(\Omega) \\ \operatorname{supp} \phi \subseteq K \end{array}$$

Proof: b) $\Rightarrow$ a)

a) $\Rightarrow$ b) Assume b) is wrong.

$\Rightarrow \exists K \subset\subset \Omega \ \exists (\phi_k)_k$ with $\operatorname{supp} \phi_k \subseteq K$ s.t.

$$|T(\phi_k)| > k \, \|\phi_k\|_k \, , \quad k = 1, 2, 3, \dots$$

We may assume $T(\phi_k) = 1$ (otherwise we substitute ϕ_k by $\psi_k = \frac{\phi_k}{T(\phi_k)}$)

Then given arbitrary j

$$\|\phi_k\|_j \overset{k \geq j}{\leq} \|\phi_k\|_k < \frac{1}{k} \xrightarrow{k \to +\infty} 0$$

$$\Rightarrow \phi_k \longrightarrow 0 \quad \text{in } \mathcal{D}(\Omega)$$

$$\text{but } T(\phi_k) = 1 \not\to 0 \ \lightning \quad (T \in \mathcal{D}') \qquad \blacksquare$$

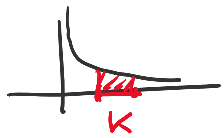
<u>Def</u> If j can be chosen independent of K, the smallest such j is called the <u>order</u> of T (and T has finite order).

<u>Example</u> "Functions are distributions"

Write $u \in L^1_{loc}(\Omega)$ if $u : \Omega \to \mathbb{C}$ measurable and

$$\int_K |u(x)| \, dx < +\infty \qquad \forall K \subset\subset \Omega$$

Example: $u(x) = \frac{1}{x} : \mathbb{R}_+ \to \mathbb{C}$



$$\int_0^{+\infty} \frac{1}{x} \, dx = +\infty \quad \text{but} \quad \int_K \frac{1}{x} \, dx < +\infty \quad K \subset\subset \mathbb{R}_+$$

Define

$$T_u(\phi) := \int_\Omega u(x) \, \phi(x) \, dx \quad , \ \phi \in \mathcal{D}(\Omega)$$

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Then $T_u \in \mathcal{D}'(\Omega)$: Let $\operatorname{supp}\phi \subseteq K$

$$|T_u(\phi)| = \left| \int_K u(x)\phi(x)dx \right|$$

$$\leq \int_K |u(x)| \underbrace{|\phi(x)|}_{\leq \|\phi\|_0} dx$$

$$\leq C_K \|\phi\|_0 \quad , C_K = \int_K |u|$$

$\Rightarrow T_u$ has order 0

Such distr. are called <u>regular distributions</u>.

One can show

$$T_u = T_v \iff u = v \quad \text{almost everywhere}$$

<u>Example</u> $\delta : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathbb{C}$

$$\delta(\phi) = \phi(0) \quad , \phi \in \mathcal{D}(\mathbb{R}^n)$$

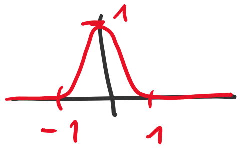
$$|\delta(\phi)| = |\phi(0)| \leq \|\phi\|_0 \quad \forall \phi$$

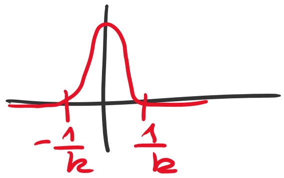
δ has order 0. δ is not regular :

suppose $u \in L^1_{loc}(\mathbb{R}^n)$ s.t. $T_u = \delta$ i.e.

$$\phi(0) = \int_{\mathbb{R}^n} u(x)\,\phi(x)\,dx \qquad \forall \phi \in \mathcal{D}(\mathbb{R}^n)$$

Let ψ be as above :





Let $\phi_k(x) := \psi(kx)$. Then

$$1 = \phi_k(0) = \int \underbrace{u(x)\,\phi_k(x)}\,dx \xrightarrow{k \to +\infty} 0 \quad \lightning$$

- $\longrightarrow 0 \quad \forall x$
- $|\ldots| \leq C\,|u|$

Lebesgue dominated conv. theorem