Lecture 30

30 November 2021 09:03

$T : \underbrace{\mathcal{D}(\Omega)}_{\uparrow} \to \mathbb{C}$ linear & cont.

C^∞ functions with compact support in Ω

$$\|\phi\|_j = \max_{\substack{|\alpha|\le j \\ x \in \mathbb{R}^n}} |\partial_x^\alpha \phi(x)| \quad , j = 0, 1, \dots$$

<u>Ex</u> $u \in L^1_{loc}(\Omega)$

"density"

$$T_u(\phi) = \int_\Omega u(x)\phi(x)\,dx \qquad , \phi \in \mathcal{D}(\Omega)$$

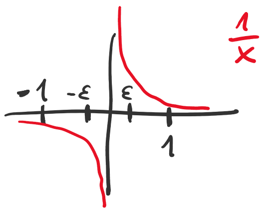
$\Rightarrow T_u \in \mathcal{D}'(\Omega)$ regular distribution

$T_u = T_v \iff u = v$ in $L^1_{loc}(\Omega)$

(i.e. $u = v$ a.e.)

$\longrightarrow$ "generalized functions"

<u>Ex</u> $\delta_{x_0}(\phi) = \phi(x_0)$, $x_0 \in \mathbb{R}$ $\rightsquigarrow$ $\delta_{x_0} \in \mathcal{D}'(\mathbb{R}^n)$



<u>Example</u> $\frac{1}{x} \notin L^1_{loc}(\mathbb{R})$

$$T(\phi) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{\phi(x)}{x}\,dx \qquad , \phi \in \mathcal{D}(\mathbb{R}).$$

Taylor expansion: $\phi(x) = \phi(0) + x\, r_\phi(x)$, $r_\phi \in C^\infty(\mathbb{R})$

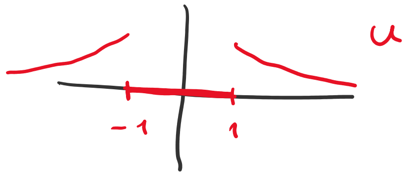
$$\left[\, r_\phi(x) = \int_0^1 \phi'(tx)\, dt \,\right]$$

$$T(\phi) = \int_{\mathbb{R}\setminus[-1,1]} \frac{\phi(x)}{x}\,dx + \lim_{\varepsilon\to 0^+} \int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \frac{\phi(x)}{x}\,dx$$

$$= \int_{\mathbb{R}} u(x)\,\phi(x)\,dx + \int_{-1}^{1} r_\phi(x)\,dx$$

where

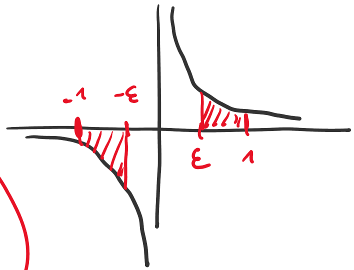
$$u(x) = \begin{cases} 0, & x\in[-1,1] \\ \frac{1}{x}, & x\in \mathbb{R}\setminus[-1,1] \end{cases}$$



$u \in L^1_{loc}(\mathbb{R})$

and because

$$\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \frac{\phi(x)}{x}\,dx = \overbrace{\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \frac{\phi(0)}{x}\,dx}^{=0} + \int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} r_\phi(x)\,dx$$



$$\xrightarrow{\varepsilon\to 0} \int_{-1}^{1} r_\phi(x)\,dx$$

$$S(\phi) := \int_{-1}^{1} r_\phi(x)\,dx$$

$$|S(\phi)| \le \int_{-1}^{1} |r_\phi(x)|\,dx$$

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$$= \int_{-1}^{1} \underbrace{\left|\frac{\phi(x)-\phi(0)}{x}\right|} dx$$

$$= |\phi'(\xi_x)| \quad \text{with } \xi_x \text{ between } 0 \text{ and } x$$

$$\leq \|\phi\|_1$$

$$\leq 2\|\phi\|_1 \qquad \forall \phi \in \mathcal{D}(\mathbb{R})$$

$\Rightarrow$ $S \in \mathcal{D}'(\mathbb{R})$ of order 1

$\Rightarrow$ $T = T_u + S \in \mathcal{D}'(\mathbb{R})$ of order 1

T is called the principal value of $\frac{1}{x}$,

$T = pv\text{-}\frac{1}{x}$.

Multiplication with smooth functions

$u \in L^1_{loc}(\Omega)$, $a \in C^\infty(\Omega)$

$\Rightarrow$ $au \in L^1_{loc}(\Omega)$

$$T_{au}(\phi) = \int_\Omega a(x)u(x)\phi(x)\, dx$$

$$= \int_\Omega u(x)\, a(x)\phi(x)\, dx$$

$$= T_u(a\phi)$$

note : $\phi \in \mathcal{D}(\Omega) \Rightarrow a\phi \in \mathcal{D}(\Omega)$

note: $\phi \in \mathcal{D}(\Omega) \implies a\phi \in \mathcal{D}(\Omega)$

<u>Definition</u> $T \in \mathcal{D}'(\Omega)$, $a \in C^\infty(\Omega)$. Then

$$\boxed{(aT)(\phi) := T(a\phi) \quad , \phi \in \mathcal{D}(\Omega)}$$

defines $aT \in \mathcal{D}'(\Omega)$.

Differentiation of distributions

$u \in C^1(\mathbb{R}) \implies u, u' \in C(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R})$

$$T_{u'}(\phi) = \int_{-\infty}^{+\infty} u'(x)\,\phi(x)\,dx$$

$$\overset{\text{int. by parts}}{=} \underbrace{u(x)\phi(x)\Big|_{x=-\infty}^{x=+\infty}}_{=0} - \int_{-\infty}^{+\infty} u(x)\,\phi'(x)\,dx$$

$$= -\int_{-\infty}^{+\infty} u(x)\phi'(x)\,dx$$

$$= -T_u(\phi')$$

<u>Definition</u> Let $T \in \mathcal{D}'(\Omega)$, $\alpha \in \mathbb{N}_0^n$. Then

$$\boxed{(\partial^\alpha T)(\phi) = (-1)^{|\alpha|}\, T(\partial^\alpha \phi)}$$

defines $\partial^\alpha T \in \mathcal{D}'(\Omega)$.

General principle Assume $A : \mathcal{D}(\Omega) \longrightarrow \mathcal{D}(\Omega)$

If $u \in \mathcal{D}(\Omega)$ and

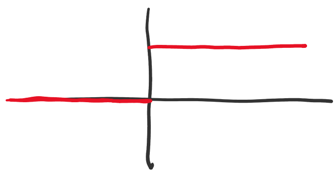
$$T_{Au}(\phi) = \ldots\ldots = T_u(A^* \phi)$$

then define for $T \in \mathcal{D}'(\Omega)$

$$\boxed{(AT)(\phi) := T(A^* \phi)}$$

Check if then $AT \in \mathcal{D}'(\Omega)$. (This is the case if $A^* : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is continuous in the following sense: $\phi_k \to \phi$ in $\mathcal{D}(\Omega)$ implies $A^* \phi_k \longrightarrow A^* \phi$ in $\mathcal{D}(\Omega)$).

__Example__
$$h(x) = \begin{cases} 0 & : x < 0 \\ 1 & : x > 0 \end{cases}$$



$$T_h(\phi) = \int_{-\infty}^{+\infty} h(x)\, \phi(x)\, dx = \int_0^{+\infty} \phi(x)\, dx$$

$$(T_h)'(\phi) \overset{\text{def.}}{=} - T_h(\phi') = - \int_0^{+\infty} \phi'(x)\, dx$$

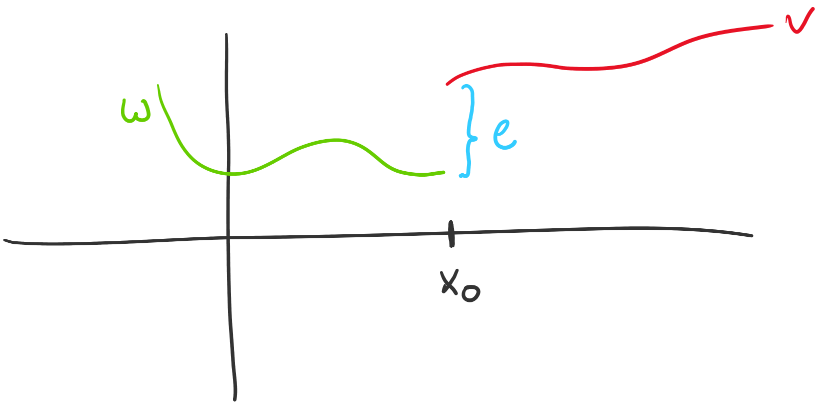
$$= - \phi(x) \Big|_{x=0}^{x=+\infty} = \phi(0)$$

$$= \delta(\phi)$$

The derivative of T_h is the δ-distribution

Theorem Let $u \in L^1_{loc}(\mathbb{R})$ be of the form

$$u(x) = \begin{cases} v(x) & , x > x_0 \\ w(x) & , x < x_0 \end{cases} \quad \text{with} \quad \begin{array}{l} v \in C^1([x_0, +\infty)), \\ w \in C^1((-\infty, x_0]) \end{array}$$



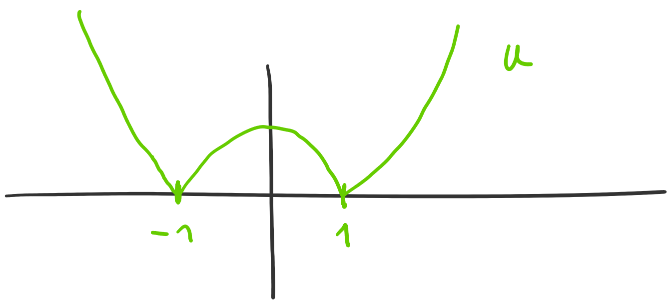
Then

$$(T_u)' = T_{\tilde{u}} + \overbrace{(v(x_0) - w(x_0))}^{e} \delta_{x_0}$$

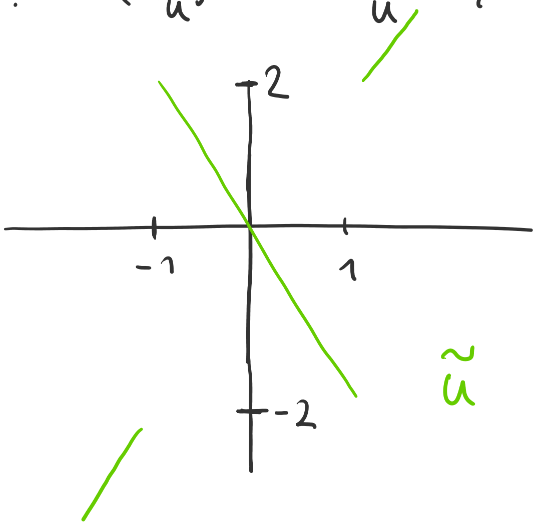
$$\text{with} \quad \tilde{u}(x) = \begin{cases} v'(x) & , x > x_0 \\ w'(x) & , x < 0 \end{cases}$$

Example

$$u(x) = |x^2 - 1| = \begin{cases} 1 - x^2 & , |x| < 1 \\ x^2 - 1 & , |x| > 1 \end{cases}$$



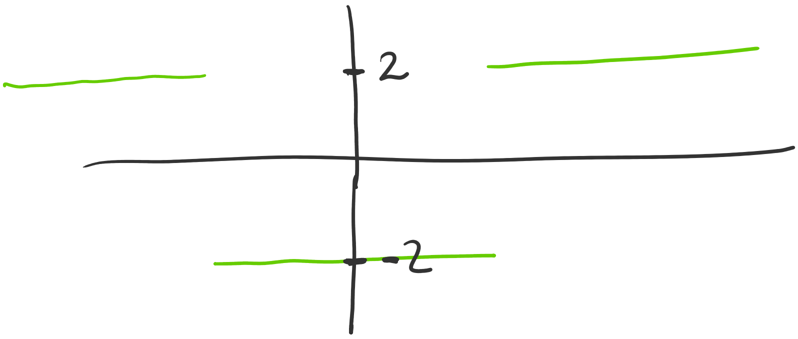
First derivative : $(T_u)' = T_{\tilde u}$, $\tilde u(x) = \begin{cases} -2x & , |x|<1 \\ 2x & , |x|>1 \end{cases}$



$\tilde u$

Second derivative : $(T_u)'' = (T_{\tilde u})' = T_{\tilde{\tilde u}} + 4\delta_{-1} + 4\delta_1$

$$\tilde{\tilde u}(x) = \begin{cases} -2 & , |x|<1 \\ 2 & , |x|>1 \end{cases}$$



$\tilde{\tilde u}$

Third derivative : $(T_u)''' = \underbrace{(T_{\tilde{\tilde u}})'}_{} + 4\delta_{-1}' + 4\delta_1'$

$$= -4\delta_{-1} + 4\delta_1$$

$$= 4(\delta_1 + \delta_1' - \delta_{-1} + \delta_{-1}')$$

$$\left[\tilde{\tilde{\tilde{u}}} = 0\right]$$

Convolution $f, g : \mathbb{R}^n \to \mathbb{C}$

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

$$= \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy$$

Theorem $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$.

If r with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ then

$f * g \in L^r(\mathbb{R}^n)$, $\|f * g\|_r \leq \|f\|_p \|g\|_q$

In part: $p = q = 1 \Rightarrow r = 1$

Sketch: $p = q = 1 = r$

$$\int |f * g|(x)\, dx = \int \left| \int f(x-y)\, g(y)\, dy \right| dx$$

$$\leq \int \int |f(x-y)|\, |g(y)|\, dy\, dx$$

$$\overset{\text{Fubini}}{=} \int \int |f(x-y)|\, |g(y)|\, dx\, dy$$

$$= \int \Big(\underbrace{\int |f(x-y)|\, dx}_{} \Big) |g(y)|\, dy$$

$$\underbrace{\int |f(x-y)|\, dx} \overset{z=x-y}{=} \int |f(z)|\, dz = \|f\|_1$$

$$= \|f\|_1 \int |g(y)|\, dy$$

$$= \|f\|_1 \, \|g\|_1$$

Extension to distributions:

$u \in \mathcal{D}(\mathbb{R}^n)$, $\phi \in \mathcal{D}(\mathbb{R}^n)$

$$(u * \phi)(x) = \int u(y) \underbrace{\phi(x-y)}_{\psi_x(y),\ \psi_x \in \mathcal{D}(\Omega)} dy =$$

$$= \int u(y)\, \psi_x(y)\, dy$$

$$= T_u(\psi_x)$$

$$\boxed{\begin{aligned} \psi_x(y) &= \phi(x-y) \\ \psi_x &= \phi(x-\bullet) \end{aligned}}$$

Definition Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\boxed{f(x) := T(\phi(x-\bullet)), \quad x \in \mathbb{R}^n,}$$

defines $f \in C^\infty(\mathbb{R}^n)$ with

$$\partial^\alpha f(x) = T\big((\partial^\alpha \phi)(x-\bullet)\big)$$

$$= (\partial^\alpha T)\big(\phi(x-\bullet)\big)$$

$T * \phi := f$ convolution of T and ϕ.

In short:

$$\boxed{\partial^\alpha (T * \phi) = (\partial^\alpha T) * \phi = T * \partial^\alpha \phi}$$

Example $\delta \in \mathcal{D}(\mathbb{R}^n)$

$$(\delta * \phi)(x) = \delta(\phi(x - \cdot))$$

$$= \phi(x - 0) = \phi(x)$$

$\Rightarrow$ $\delta * \phi = \phi$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$.