Lecture 31a

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$T \in \mathcal{D}'(\Omega)$

- $a \in C^\infty(\Omega)$. Then $aT \in \mathcal{D}'(\Omega)$ with

$$\boxed{(aT)(\phi) = T(a\phi) \quad \forall \phi \in \mathcal{D}(\Omega)}$$

- $\partial^\alpha T \in \mathcal{D}'(\Omega)$ with

$$\boxed{(\partial^\alpha T)(\phi) = (-1)^{|\alpha|} T(\partial^\alpha \phi) \quad \forall \phi \in \mathcal{D}(\Omega)}$$

- $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$\boxed{(T * \phi)(x) = T\big(\phi(x - \cdot)\big), \quad x \in \mathbb{R}^n}$$

$T * \phi \in C^\infty(\mathbb{R}^n)$ and

$$\boxed{\partial^\alpha (T * \phi) = (\partial^\alpha T) * \phi = T * (\partial^\alpha \phi)}$$

Distributions and partial differential equations

ordinary diff. eq: $\quad x'(t) = f(t, x(t))$

$t \in \mathbb{R}$

Differential operator with constant coefficients

$$A = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha, \quad a_\alpha \in \mathbb{C}$$

"coefficients"

$A : \mathcal{D}'(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$

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$$AT = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha T$$

Explicitly:

$$AT(\phi) = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha T(\phi)$$

$$= \sum_{|\alpha|\le m} a_\alpha (-1)^{|\alpha|} T(\partial^\alpha \phi)$$

$$= T\Big(\underbrace{\sum_{|\alpha|\le m} (-1)^{|\alpha|} a_\alpha \partial^\alpha}_{=: A^t} \phi \Big)$$

$$= T(A^t \phi)$$

Examples
- $\Delta = \partial^2_{x_1} + \dots + \partial^2_{x_n}$ Laplace operator in $\mathbb{R}^n$
- $\partial_t - \Delta$ in $\mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}$ heat operator
- $\partial_t^2 - \Delta$ —— " —— wave operator

We are interested in solving pde: Given $S \in \mathcal{D}'(\mathbb{R}^n)$ find solution $T \in \mathcal{D}'(\mathbb{R}^n)$ s.t.

$$\boxed{AT = S} \iff AT(\phi) = S(\phi) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n)$$

$\Longleftrightarrow \quad T(A^t\phi) = S(\phi) \quad$ —— '' ——

<u>Definition</u> A distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a fundamental solution of A if

$$AE = \delta \iff E(A^t\phi) = \phi(0) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n)$$

Application: Given $\phi \in \mathcal{D}(\mathbb{R}^n)$ let $u := E * \phi$. Then

$$\begin{aligned} A(E*\phi) &= \sum_{|\alpha|\le m} a_\alpha \partial^\alpha (E*\phi) \\ &= \sum_{|\alpha|\le m} a_\alpha (\partial^\alpha E) * \phi \\ &= \Big(\sum_{|\alpha|\le m} a_\alpha \partial^\alpha E\Big) * \phi \\ &= (AE) * \phi \\ &= \delta * \phi \\ &= \phi \end{aligned}$$

$$(\alpha T + \beta S) * \phi = \alpha\, T*\phi + \beta\, S*\phi$$

$\Rightarrow$ $\boxed{u = E*\phi \text{ is a solution of } Au = \phi}$

<u>Malgrange-Ehrenpreis Theorem</u> Every $A \neq 0$ with constant coefficients has a fundamental solution.

<u>Example</u> Δ has fundamental solution $E = T_e$ with

$\underline{n=2}$: $\quad e(x) = \frac{1}{2\pi} \log|x| \qquad (x=(x_1,x_2))$

$\underline{n\geq 3}$: $\quad e(x) = \frac{\Gamma(n/2)}{(2-n)\, 2\pi^{n/2}} \; \frac{1}{|x|^{n-2}} \qquad (x=(x_1,\dots,x_n))$

In any case $e \in C^\infty(\mathbb{R}^n \setminus 0) \cap L^1(\mathbb{R}^n)$.

For $\phi \in \mathcal{D}(\mathbb{R}^n)$ a solution of $\Delta u = \phi$ is

$$u(x) = (e*\phi)(x)$$

$$\boxed{u(x) \overset{n\geq 3}{=} \frac{\Gamma(n/2)}{(2-n)\, 2\pi^{n/2}} \int_{\mathbb{R}^n} \frac{\phi(x)}{|x-y|^{n-2}}\, dy}$$

$\underline{\text{Example}}$ Let $A = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$ with $a\neq 0$.

(second order $(m=2)$ diff. op. on $\mathbb{R}$).

Let $v \in C^\infty(\mathbb{R})$ be the unique solution of

$$(1) \quad \left.\begin{aligned} & av'' + bv' + c = 0 \quad \text{on } \mathbb{R}, \\ & v(0) = 0, \quad v'(0) = \frac{1}{a}. \end{aligned}\right\}$$

[$p(\lambda) = a\lambda^2 + b\lambda + c = a(\lambda-\lambda_0)(\lambda-\lambda_1)$

- $\lambda_0 \neq \lambda_1$: The solutions to (1) are

$$v(x) = c_0 e^{\lambda_0 x} + c_1 e^{\lambda_1 x}, \quad c_0, c_1 \in \mathbb{C}$$

- $\lambda_0 = \lambda_1$: $\quad v(x) = c_0 e^{\lambda_0 x} + c_1 x e^{\lambda_0 x}, \quad c_0, c_1 \in \mathbb{C}$

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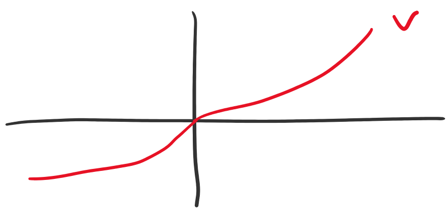
Define

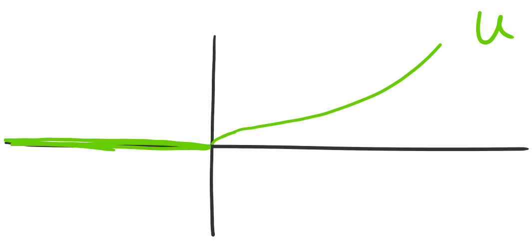
$$u(x) = \begin{cases} v(x) & , x > 0 \\ 0 & , x < 0 \end{cases}.$$

Then $E = T_u$ is a fundamental solution of A.

PROOF

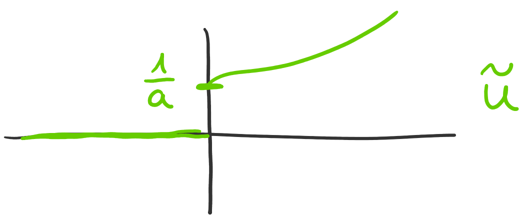


v is C^∞



u is C

$$(T_u)' = T_{\tilde u} + 0 \cdot \delta, \quad \tilde u(x) = \begin{cases} v'(x) & , x > 0 \\ 0 & , x < 0 \end{cases}$$



$$(T_u)'' = (T_{\tilde u})' = T_{\tilde{\tilde u}} + \frac{1}{a}\delta, \quad \tilde{\tilde u}(x) = \begin{cases} v''(x), & x > 0 \\ 0, & x < 0 \end{cases}$$

$$A T_u = a (T_u)'' + b (T_u)' + c T_u$$

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$$= a\left(T_{\tilde{\tilde{u}}} + \frac{1}{a}\delta\right) + b\, T_{\tilde{u}} + c\, T_u$$

$$= T_{a\tilde{\tilde{u}}} + \delta + T_{b\tilde{u}} + T_{cu}$$

$$= T_{a\tilde{\tilde{u}} + b\tilde{u} + cu} + \delta$$

$$(a\tilde{\tilde{u}} + b\tilde{u} + cu)(x) = \begin{cases} \underbrace{av'' + bv' + cv}_{=0} & , x > 0 \\ 0 & , x < 0 \end{cases}$$

$$= 0$$

$$\Rightarrow \underline{A T_u = \delta}$$