The Fourier transform

The Fourier transform on L^1-functions

$f \in L^1(\mathbb{R}^n)$ $\qquad$ $\mathcal{F}f : \mathbb{R}^n \longrightarrow \mathbb{C}$

$$\boxed{(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx \ , \quad \xi \in \mathbb{R}^n.}$$

$$x\xi = x \cdot \xi = (x_1, \dots, x_n) \cdot (\xi_1, \dots, \xi_n)$$

$$= x_1\xi_1 + x_2\xi_2 + \dots + x_n\xi_n$$

[There are other definitions of $\mathcal{F}$, e.g.

$$(\mathcal{F}f)(\xi) = (2\pi)^{-n/2} \int \dots ,$$

$$(\mathcal{F}f)(\xi) = \int e^{-2\pi i x\xi} f(x)\,dx \qquad]$$

Lemma The following is true:

a) $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ linear and bounded

with $\|\mathcal{F}\|_{\mathcal{L}(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))} \leq 1$.

b) If $f \in L^1(\mathbb{R}^n)$ then $\mathcal{F}f \in C(\mathbb{R}^n)$ (continuous)

and $\lim_{|\xi| \to +\infty} |\mathcal{F}f(\xi)| = 0$. "Riemann-Lebesgue Lemma" and $\boxed{\lim_{|\xi|\to+\infty} |\mathcal{F}f(\xi)| = 0.}$ Riemann-Lebesgue Lemma" without proof

PROOF a) For every $\xi \in \mathbb{R}^n$

$$|\mathcal{F}f(\xi)| \leq \int |e^{-ix\xi} f(x)|\,dx$$

$$= \int |f(x)|\,dx$$

$$= \|f\|_{L^1(\mathbb{R}^n)}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$|e^{i\theta}| = \cos^2 + \sin^2 = 1$

$$\Rightarrow \quad \|\mathcal{F}f\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \qquad \forall f \in L^1(\mathbb{R}^n)$$

$$\mathcal{F}(f+g)(\xi) = \int e^{-ix\xi}(f(x)+g(x))\,dx$$

$$= \int e^{-ix\xi} f(x)\,dx + \int e^{-ix\xi} g(x)\,dx$$

$$= \mathcal{F}f(\xi) + \mathcal{F}g(\xi)$$

$$\mathcal{F}(\alpha f)(\xi) = \int e^{-ix\xi}\alpha f(x)\,dx = \alpha \int e^{-ix\xi} f(x)\,dx$$

$$= \alpha(\mathcal{F}f)(\xi)$$

$\Rightarrow$ $\mathcal{F}$ is linear.

b) Let $\xi_k \longrightarrow \xi$ in $\mathbb{R}^n$.

We have to show: $\mathcal{F}f(\xi_k) \longrightarrow \mathcal{F}f(\xi)$

Lebesgue dominated convergence theorem: If

i) $g_k(x) \xrightarrow{k\to+\infty} g(x)$ for a. every $x \in \mathbb{R}^n$

ii) $\exists\, G \in L^1(\mathbb{R}^n)$: $|g_k(x)| \leq G(x)$ $\forall k$, a. every x

$$\Rightarrow \lim_{k\to+\infty} \int g_k(x)\,dx = \int g(x)\,dx .$$

$$\mathcal{F}f(\xi_k) = \int \underbrace{e^{-ix\xi_k} f(x)}_{=:\, g_k(x)}\,dx$$

Then

i) $g_k(x) = e^{-ix\xi_k} f(x) \xrightarrow{k\to+\infty} e^{-ix\xi} f(x) =: g(x)$ $\forall x$

ii) $|g_k(x)| \leq |f(x)| =: G(x)$

Theorem $\Rightarrow$

$$\mathcal{F}f(\xi_k) \xrightarrow{k\to+\infty} \int g(x)\,dx = \int e^{-ix\xi} f(x)\,dx$$

$$= (\mathcal{F}f)(\xi).$$

$\Rightarrow \mathcal{F}f \in C(\mathbb{R}^n)$.