Lezione 32

02 December 2021 09:02

$f \in L^1(\mathbb{R}^n)$

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int e^{-ix\xi} f(x)\, dx \quad , \ \xi \in \mathbb{R}^n$$

$\hat{f} : \mathbb{R}^n \to \mathbb{C}$

$\mathcal{F} : L^1 \to L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$

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**Lemma** $f, g \in L^1(\mathbb{R}^n)$. Then:

a) $\widehat{f*g} = \hat{f}\,\hat{g}$

b) $\int \hat{f}(\xi) g(\xi)\, d\xi = \int f(\xi) \hat{g}(\xi)\, d\xi$

**PROOF**:

a) $$\widehat{f*g}(\xi) = \int e^{-ix\xi} (f*g)(x)\, dx$$

$$= \int \underbrace{e^{-ix\xi}}\left(\int f(x-y) g(y)\, dy\right) dx$$

$$= e^{-i(x-y+y)\xi} = e^{-i(x-y)\xi} e^{-iy\xi}$$

$$\overset{\text{Fubini}}{=} \int e^{-iy\xi} g(y) \underbrace{\left(\int e^{-i(x-y)\xi} f(x-y)\, dx\right)} dy$$

$$\overset{z=x-y}{=} \int e^{-iz\xi} f(z)\, dz = \hat{f}(\xi)$$

$$= \hat{f}(\xi)\,\hat{g}(\xi)$$

b) $\int \hat{f}(\xi)\, g(\xi)\, d\xi = \int \left( \int e^{-ix\xi} f(x)\, dx \right) g(\xi)\, d\xi$

$$\overset{\text{Fubini}}{=} \int \underbrace{\left( \int e^{-ix\xi} g(\xi)\, d\xi \right)}_{= \hat{g}(x)} f(x)\, dx$$

$$= \int \hat{g}(x) f(x)\, dx = \int \hat{g}(\xi) f(\xi)\, d\xi$$

<u>Remark</u> If $u \in L^1$, $\phi \in \mathcal{D}(\mathbb{R}^n)$ then

$$T_{\hat{u}}(\phi) = \int \hat{u}(\xi)\, \phi(\xi)\, d\xi = \int u(\xi)\, \hat{\phi}(\xi)\, d\xi = T_u(\hat{\phi})$$

Suggests to define for $T \in \mathcal{D}'(\mathbb{R}^n)$

$$\hat{T}(\phi) := T(\hat{\phi}) \quad , \phi \in \mathcal{D}(\mathbb{R}^n).$$

"Unfortunately"

$$\phi \in \mathcal{D}(\mathbb{R}^n) \Rightarrow \hat{\phi} \notin \mathcal{D}(\mathbb{R}^n) \qquad (\text{see later})$$

Solution: Substitute $\mathcal{D}(\mathbb{R}^n)$ by another space:

## Reapidly decreasing functions – $\mathcal{S}(\mathbb{R}^n)$

<u>Definition</u> $\mathcal{S}(\mathbb{R}^n)$ space of all $\varphi \in C^\infty(\mathbb{R}^n)$ with

$$\boxed{\|\varphi\|_{(N)} := \sup_{\substack{|\alpha|+|\beta| \le N, \\ x \in \mathbb{R}^n}} \left| x^\beta \partial_x^\alpha \varphi(x) \right| < +\infty \quad \forall N \in \mathbb{N}_0}$$

$$x^\beta = x_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n}$$

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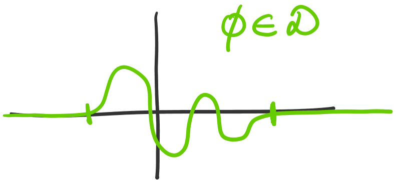


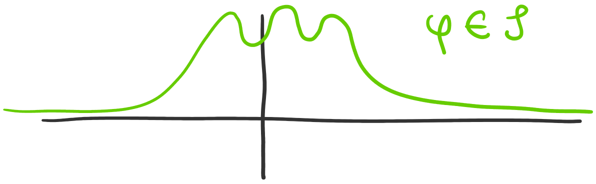


__Example__: $\varphi(x) = e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$

$(\varphi_k)_k \subseteq \mathcal{S}(\mathbb{R}^n)$, then $\varphi_k \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$ $:\Longleftrightarrow$

$$\boxed{\|\varphi_k - \varphi\|_{(N)} \xrightarrow{k \to +\infty} 0 \quad \forall N \in \mathbb{N}_0}$$

__Remark__

$$d(\varphi, \psi) = \sum_{k=0}^{+\infty} \frac{1}{2^k} \underbrace{\frac{\|\varphi - \psi\|_k}{1 + \|\varphi - \psi\|_k}}_{\leq 1}, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$$

defines a metric on $\mathcal{S}(\mathbb{R}^n)$. Then convergence in $(\mathcal{S}, d)$ coincides with the above convergence.
Moreover $(\mathcal{S}, d)$ is a complete metric space.
( $\mathcal{S}$ is a "Fréchet space").

__Remark__ $\varphi, \psi \in \mathcal{S} \Rightarrow \varphi + \psi \in \mathcal{S}$, $\alpha\varphi \in \mathcal{S}$ $\forall \alpha \in \mathbb{C}$

$\mathcal{S}$ is a vector space (over $\mathbb{C}$).

__Lemma__ $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$, $1 \leq p \leq +\infty$

__PROOF__ __$p = +\infty$__: $\|\varphi\|_{(0)} = \sup_{x \in \mathbb{R}^n} |\varphi(x)| = \|\varphi\|_{\infty, \mathbb{R}^n}$

PROOF $p=+\infty$: $\|\varphi\|_{(0)} = \sup_{x\in\mathbb{R}^n} |\varphi(x)| = \|\varphi\|_{L^\infty(\mathbb{R}^n)}$

$1 \le p < +\infty$:

$$\|\varphi\|_{L^p}^p = \int_{|x|\le 1} |\varphi(x)|^p dx +$$

$$+ \int_{|x|\ge 1} |x|^{-2Np} \left| |x|^{2N} \varphi(x) \right|^p dx = I_1(\varphi) + I_2(\varphi)$$

$$I_1(\varphi) \le \int_{|x|\le 1} \|\varphi\|_{(0)}^p \, dx = \underbrace{\left( \int_{|x|\le 1} dx \right)}_{=:\tau_n \text{ volume of the unit-ball in } \mathbb{R}^n} \cdot \|\varphi\|_{(0)}^p$$

$$|x|^{2N} = (x_1^2 + \dots + x_n^2)^N = \sum_{|\beta|\le 2N} c_\beta x^\beta \quad (c_\beta \text{ appropriate})$$

$$I_2(\varphi) \le c_N \int_{|x|\ge 1} |x|^{-2Np} dx \quad \|\varphi\|_{(2N)}^p$$

Using polar-coordinates

$$\int_{|x|\ge 1} |x|^{-2Np} dx = n\tau_n \int_1^{+\infty} r^{-2Np+n-1} dr$$

$\Big[$ In $\mathbb{R}^2$:
$$\int_{\mathbb{R}^2} f(x) dx = \int_0^{2\pi} \int_0^{+\infty} f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta$$

$$f(x) = |x|^\alpha \Rightarrow f(r\cos\theta, \sin\theta) = r^\alpha \quad \Big]$$

$$\int_1^{+\infty} r^\alpha \, dr < +\infty \iff \alpha < -1$$

Choose $N$ s.t.

$$-2Np + n - 1 < -1 \iff N > \frac{n}{2p}$$

Summarizing:

$$\boxed{\|\varphi\|_{L^p} \le C_N \|\varphi\|_{(2N)} \quad \forall \varphi \in \mathcal{S} \quad \forall N > \frac{n}{2p}} \qquad (*)$$

<u>Theorem</u> If $1 \le p < +\infty$ then $\mathcal{D}(\mathbb{R}^n)$ is a dense subset of $L_p(\mathbb{R}^n)$, i.e.

$$\forall f \in L^p(\mathbb{R}^n) \quad \exists (\phi_k)_k \subseteq \mathcal{D}(\mathbb{R}^n) :$$

$$\|f - \phi_k\|_{L^p} \xrightarrow{k \to +\infty} 0$$

<u>Definition</u> Let $X$ be either a Banach space or $X = \mathcal{S}(\mathbb{R}^n)$. A linear map $T : \mathcal{S}(\mathbb{R}^n) \longrightarrow X$ is said to be continuous if for every convergent sequence $(\varphi_k)_k \subseteq \mathcal{S}(\mathbb{R}^n)$:

$$\boxed{\lim_{k \to +\infty} T(\varphi_k) = T\left(\lim_{k \to +\infty} \varphi_k\right)}$$

$$\left[\ \underbrace{\varphi_k \to \varphi}_{\text{in } \mathcal{S}} \implies \underbrace{T(\varphi_k) \longrightarrow T(\varphi)}_{\text{in } X}\ \right]$$

$$\underbrace{\qquad\qquad}_{\text{in } \mathcal{S}} \qquad\qquad \underbrace{\qquad\qquad\qquad\qquad}_{\text{in } X}$$

<u>Example</u> $T(\varphi) = \varphi$ then $T: \mathcal{S}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$

continuous because

$$\| T(\varphi) \|_p \leq C \, \|\varphi\|_{(2N)}$$

Thus if $\varphi_k \longrightarrow \varphi$ in $\mathcal{S}$ then

$$\| T(\varphi_k) - T(\varphi) \|_p = \| T(\varphi_k - \varphi) \|_p$$

$$\leq C \, \| \varphi_k - \varphi \|_{(2N)} \xrightarrow{k \to +\infty} 0$$

<u>Theorem</u> • For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$ :

$$\boxed{\begin{aligned} \widehat{\partial^\alpha \varphi}(\xi) &= i^{|\alpha|} \xi^\alpha \hat{\varphi}(\xi) \\ (-i)^{|\alpha|} \widehat{x^\alpha \varphi}(\xi) &= \partial_\xi^\alpha \hat{\varphi}(\xi) \end{aligned}} \qquad (\xi \in \mathbb{R}^n)$$

• $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ linear and continuous.

PROOF:

$$\widehat{\partial_{x_j} \varphi}(\xi) = \int e^{-ix\xi} (\partial_{x_j} \varphi)(x) \, dx$$

$$\boxed{\begin{aligned} x &= (x_1, \dots, x_n) \\ &= (x_j, x') \end{aligned}} \quad = \int_{\mathbb{R}^{n-1}} e^{-ix'\xi'} \underbrace{\left( \int_{\mathbb{R}} e^{-ix_j \xi_j} (\partial_{x_j} \varphi)(x_j, x') \, dx_j \right)}_{\text{integration by parts:}} dx'$$

$$= e^{-ix_j \xi_j} \cancel{\varphi(x_j, x') \Big|^{x_j = +\infty}} \; 0 \; - \; [illegible]$$

$$= \cancel{e^{-ix_j\xi_j}\varphi(x_j,x')\Big|_{x_j=-\infty}^{x_j=+\infty}} \; 0 \; - \int_{\mathbb{R}} -i\xi_j\, e^{-ix_j\xi_j}\varphi(x_j,x')\,dx_j$$

$$= i\xi_j \int e^{-ix\xi}\varphi(x)\,dx$$

$$\widehat{\partial_{x_j}\varphi}(\xi) = i\xi_j\,\hat{\varphi}(\xi)$$

$$\widehat{\partial_{x_j}\partial_{x_k}\varphi}(\xi) = i\xi_j\,\widehat{\partial_{x_k}\varphi}(\xi) = i^2\xi_j\xi_k\,\hat{\varphi}(\xi)$$

By induction: $\widehat{\partial^\alpha\varphi}(\xi) = i^{|\alpha|}\xi^\alpha\hat{\varphi}(\xi)$

Similarly:

$$\partial_{\xi_j}\hat{\varphi}(\xi) = \partial_{\xi_j}\int e^{-ix\xi}\varphi(x)\,dx$$

$$= \int \underbrace{\left(\partial_{\xi_j} e^{-ix\xi}\right)}\varphi(x)\,dx$$

$$\partial_{\xi_j} e^{-i(x_1\xi_1+\dots+x_n\xi_n)} = -ix_j\,e^{-ix\xi}$$

$$= -i\int e^{-ix\xi}(x_j\varphi)(x)\,dx$$

$$= -i\,\widehat{x_j\varphi}(\xi)$$

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$$|\hat{\varphi}(\xi)| \le \int |\varphi(x)|\,dx = \|\varphi\|_{L^1(\mathbb{R}^n)}$$

$$\overset{(*)}{\le} C\,\|\varphi\|_{(\ldots)} \qquad \forall\,\xi\in\mathbb{R}^n$$

$$\overset{(*)}{\leq} C \|\varphi\|_{(n+2)} \qquad \forall \xi \in \mathbb{R}^n$$

$$\Rightarrow \quad \|\hat{\varphi}\|_{(0)} \leq C \|\varphi\|_{(n+2)}$$

By induction, using the previous rules,

$$\boxed{\|\hat{\varphi}\|_{(N)} \leq C_N \|\varphi\|_{(N+n+2)} \qquad \forall \varphi \in \mathcal{S} \;\; \forall N}$$

$$\Big[\text{ e.g. } \quad \|\hat{\varphi}\|_{(1)} = \max_{j=1}^{n} \underbrace{\|x_j \hat{\varphi}\|_{(0)}}_{} + \|\partial_{x_j} \hat{\varphi}\|_{(0)}$$

$$= \|\widehat{\partial_{x_j}\varphi}\|_{(0)} \leq C \|\partial_{x_j}\varphi\|_{(n+2)}$$

$$\leq C \|\varphi\|_{(n+2+1)} \Big]$$

Let $\varphi_k \longrightarrow \varphi$ in $\mathcal{S}$. Then

$$\|\hat{\varphi}_k - \hat{\varphi}\|_{(N)} = \|\widehat{\varphi_k - \varphi}\|_{(N)}$$

$$\leq C_N \|\varphi_k - \varphi\|_{(N+n+2)} \xrightarrow{k \to +\infty} 0$$

for every $N \in \mathbb{N}_0$. Thus $\hat{\varphi}_k \longrightarrow \hat{\varphi}$ in $\mathcal{S}$. ∎