Lezione 33

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$$\mathcal{F}f(\xi) = \int e^{-ix\xi} f(x)\,dx \quad , \quad f \in L^1(\mathbb{R}^n)$$
$$= \hat{f}(\xi)$$

$$\mathcal{F} : L^1(\mathbb{R}^n) \longrightarrow L^\infty(\mathbb{R}^n)$$

$\varphi \in \mathcal{S}(\mathbb{R}^n)$ rapidly decreasing functions $\Longleftrightarrow$

$$\|\varphi\|_{(N)} = \sup_{\substack{|\alpha|+|\beta|\le N \\ x\in\mathbb{R}^n}} |x^\beta \partial^\alpha \varphi(x)| < +\infty \quad \forall N$$

$$\varphi_k \longrightarrow \varphi \text{ in } \mathcal{S} \iff \|\varphi_k - \varphi\|_{(N)} \longrightarrow 0$$

- $\widehat{\partial^\alpha \varphi}(\xi) = i^{|\alpha|} \xi^\alpha \hat{\varphi}(\xi)$

  $(-i)^{|\alpha|} \widehat{x^\alpha \varphi}(\xi) = \partial^\alpha_\xi \hat{\varphi}(\xi)$

- $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ linear & cont.

  $\varphi_k \longrightarrow \varphi$ in $\mathcal{S}$ then $\hat{\varphi}_k \longrightarrow \hat{\varphi}$ in $\mathcal{S}$.

<u>Theorem</u> $\boxed{\mathcal{F}^2\varphi(x) = (2\pi)^n \varphi(-x) \quad \text{for all } \varphi \in \mathcal{S}.}$

Therefore $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$ is bijective with inverse

$$\boxed{(\mathcal{F}^{-1}\psi)(x) = \check{\psi}(x) = (2\pi)^{-n} \int e^{ix\xi} \psi(\xi)\,d\xi}$$

$$\dots \quad (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}} \quad (\text{Gaussian})$$

PROOF: Let $f(x) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}}$ (Gaussian).

Step 1: We show that $\hat{f}(\xi) = (2\pi)^{n/2} f(\xi)$.

n=1 (general case is a homework).

Let $u(x) = e^{-x^2/2}$. Then

$$u'(x) = -x e^{-x^2/2} = -x u(x)$$

$$u(0) = 1$$

and

$$\hat{u}'(\xi) \overset{2.7}{=} -i \widehat{xu}(\xi) \overset{-xu=u'}{=} i \widehat{u'}(\xi) = i^2 \xi \hat{u}(\xi)$$

$$= -\xi \hat{u}(\xi),$$

$$\hat{u}(0) = \int e^{-x^2/2} dx = \sqrt{2\pi}$$

$\Rightarrow$ Both u and $\hat{u}/\sqrt{2\pi}$ are solutions of the initial value problem

$$y'(t) = -t y(t), \quad y(0) = 1. \qquad (*)$$

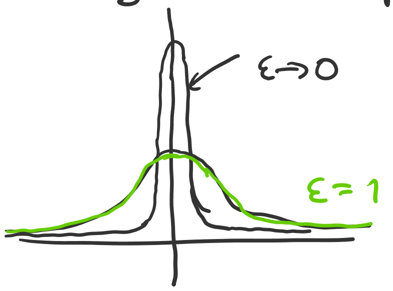
$(*)$ has a unique (maximal) solution, hence

$$u(x) = \frac{\hat{u}(x)}{\sqrt{2\pi}} \quad \forall x \in \mathbb{R}.$$

$$f = u/\sqrt{2\pi} \implies \hat{f} = \frac{\hat{u}}{\sqrt{2\pi}} = u = \sqrt{2\pi} f$$

Step 2 Let $f_\varepsilon(x) = \varepsilon^{-n} f\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$.

$\{ f_\varepsilon \mid \varepsilon > 0 \}$ is an approximate identity (note $\|f\|_{L^1} = 1$).



Moreover

$$\boxed{\widehat{f(\varepsilon \cdot)}(\xi) = (2\pi)^{n/2} f_\varepsilon(x)}$$

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$$\widehat{f(\varepsilon \cdot)}(\xi) = \int e^{-ix\xi} f(\varepsilon x)\,dx \underset{dy = \varepsilon^n dx}{\overset{y = \varepsilon x}{=}} \int e^{-i\frac{y}{\varepsilon}\xi} f(y)\,\varepsilon^{-n}\,dy$$

$$= \varepsilon^{-n} \int e^{-iy\xi/\varepsilon} f(y)\,dy$$

$$= \varepsilon^{-n} \hat{f}\left(\frac{\xi}{\varepsilon}\right)$$

$$\overset{\text{Step 1}}{=} (2\pi)^{n/2} \varepsilon^{-n} f\left(\xi/\varepsilon\right)$$

$$= (2\pi)^{n/2} f_\varepsilon(\xi)$$

$\lrcorner$

Therefore

$$(2\pi)^{-n/2} \mathcal{F}^2 \varphi(y)$$

$$= \lim_{\varepsilon \to 0} \int e^{-iy\xi} f(\varepsilon\xi)\,\hat{\varphi}(\xi)\,d\xi \qquad [f(0) = (2\pi)^{-n/2}]$$

$$\overset{(*)}{=} \lim_{\varepsilon\to 0} \int f(\varepsilon\xi)\,\mathcal{F}(\varphi(\cdot-y))(\xi)\,d\xi$$

$$= \lim_{\varepsilon\to 0} (2\pi)^{n/2} \int f_\varepsilon(x)\,\varphi(x-y)\,dx \qquad \left[\int \hat{f} g = \int f \hat{g}\right]$$

$$= (2\pi)^{n/2}\,\varphi(0-y)$$

$$= (2\pi)^{n/2}\,\varphi(-y)$$

$$\Rightarrow \mathcal{F}^2\varphi(y) = (2\pi)^n\,\varphi(-y)$$

$(*)$: $$e^{-iy\xi}\hat{\varphi}(\xi) = e^{-iy\xi}\int e^{-ix\xi}\varphi(x)\,dx$$

$$= \int e^{-i(x+y)\xi}\,\varphi(x)\,dx$$

$$\underset{dz=dx}{\overset{z=x+y}{=}} \int e^{-iz\xi}\,\varphi(z-y)\,dz$$

$$= \widehat{\varphi(\cdot-y)}(\xi)$$

<u>Step 3</u> Define $R : \mathcal{S} \longrightarrow \mathcal{S}$ by

$$(R\varphi)(x) = (2\pi)^n\,\varphi(-x)$$

Then R is bijective with inverse

$$(R^{-1}\varphi)(x) = (2\pi)^{-n}\,\varphi(-x).$$

Moreover $\mathcal{F}^2 = R$

$$\Rightarrow (R^{-1}\mathcal{F})\mathcal{F} = \mathrm{Id}.$$

$$\Rightarrow \; (\underbrace{R^{-1}\mathcal{F}}_{T})\mathcal{F} = \mathrm{Id}_{\mathcal{S}\to\mathcal{S}}$$

$\Rightarrow$ $\mathcal{F}$ has a "left-inverse"

$\Rightarrow$ $\mathcal{F}$ is injective

[$\mathcal{F}\varphi = \mathcal{F}\psi \Rightarrow$
$\varphi = T\mathcal{F}\varphi = T\mathcal{F}\psi = \psi$]

$$\Rightarrow \; \mathcal{F}(\underbrace{\mathcal{F}R^{-1}}_{S}) = \mathrm{id}_{\mathcal{S}\to\mathcal{S}}$$

$\Rightarrow$ $\mathcal{F}$ has a "right-inverse"

$\Rightarrow$ $\mathcal{F}$ is surjective

[Given $\varphi \in \mathcal{S}$ let $\psi = S\varphi$
$\Rightarrow \mathcal{F}\psi = \mathcal{F}S\varphi = \varphi$]

$\Rightarrow$ $\mathcal{F}$ is bijective with $\mathcal{F}^{-1} = R^{-1}\mathcal{F}$

[$\Rightarrow$ integral expression for $\mathcal{F}^{-1}$]

$\blacksquare$

PARSEVAL'S FORMULA

$$(\hat{\varphi}, \hat{\psi})_{L^2(\mathbb{R}^n)} = (2\pi)^n (\varphi, \psi)_{L^2(\mathbb{R}^n)}$$

$$\forall \varphi \in \mathcal{S}$$

$$\|\hat{\varphi}\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} \|\varphi\|_{L^2(\mathbb{R}^n)}$$

PROOF: $(f, g) = \int f(x)\overline{g(x)}\,dx$ L^2-inner product

$$\overline{\hat{\psi}(\xi)} = \overline{\int e^{-ix\xi}\psi(x)\,dx} = \int e^{ix\xi}\,\overline{\psi(x)}\,dx$$

$$= (2\pi)^n (\mathcal{F}^{-1}\overline{\psi})(\xi)$$

$$= (2\pi) \ (\mathcal{F}^{-1}\overline{\psi})(\xi)$$

Thus

$$(\hat\varphi,\hat\psi) = \int \hat\varphi\,\overline{\hat\psi} = \int \hat\varphi\,(\mathcal{F}^{-1}\overline{\psi}) \overset{(2\pi)^n}{=} \int \varphi\, \mathcal{F}(\mathcal{F}^{-1}\overline{\psi})$$

$$= (2\pi)^n \int \varphi\,\overline{\psi} = (2\pi)^n (\varphi,\psi)$$

$\blacksquare$

<u>PLANCHEREL'S THEOREM</u> $A := (2\pi)^{-n/2}\mathcal{F}$.

Then $A : \mathcal{S} \to \mathcal{S}$ extends to a <u>unitary isomorphism</u>

$A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, i.e.,

$$AA^* = A^*A = \mathrm{Id}_{L^2 \to L^2}.$$

<u>PROOF:</u> Plancherel $\Rightarrow$

$$(A\varphi, A\psi) = (2\pi)^{-n} (\hat\varphi,\hat\psi) = (\varphi,\psi) \quad \forall \varphi,\psi \in \mathcal{S}$$

$$\|A\varphi\|_{L^2} = \|\varphi\|_{L^2} \qquad \forall \varphi \in \mathcal{S}$$

Let $f \in L^2$.

$\Rightarrow \exists (\varphi_k) \subset \mathcal{S} : \varphi_k \longrightarrow f$ in L^2

$$\|A\varphi_k - A\varphi_\ell\|_{L^2} = \|A(\varphi_k - \varphi_\ell)\|_{L^2} = \|\varphi_k - \varphi_\ell\|_{L^2} \xrightarrow{k,\ell \to +\infty} 0$$

$\Rightarrow (A\varphi_k)_k$ is a Cauchy-sequence in L^2

$\Rightarrow (A\varphi_k)$ converges in L^2; we define

$$\boxed{Af := \lim_{k \to +\infty} A\varphi_k}$$

(homework: Independent of choice of (φ_k))

$$\|Af\|_{L^2} = \lim_k \|A\varphi_k\|_{L^2} = \lim_k \|\varphi_k\|_{L^2} = \|f\|_{L^2}$$

$\Rightarrow A \in \mathcal{L}(L^2(\mathbb{R}^n))$ with $\|A\| = 1$.

Define $B := (2\pi)^{n/2} \mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}$.

Repeat construction for A: B extends to

$B : L^2 \to L^2$. $\quad$ ($Bf = \lim_{k\to\infty} B\varphi_k$ if $\varphi_k \to f$)

Let $f \in L^2$ and $\varphi_k \to f$. Then

$$ABf = A(\lim_k \underbrace{B\varphi_k}_{\substack{\in \mathcal{S}, \\ B\varphi_k \to f}}) = \lim_k A(B\varphi_k) = \lim_k \varphi_k = f$$

$\forall f \in L^2$

$$BAf = \dots = f$$

$\Rightarrow B = A^{-1}$ as operators $L^2 \to L^2$

Let $f, g \in L^2$ and $\varphi_k \xrightarrow{\mathcal{S}} f$, $\psi_k \xrightarrow{\mathcal{S}} g$

$\Rightarrow A\varphi_k \to Af$, $A\psi_k \to Ag$

$$\Rightarrow (Af, Ag)_{L^2} = \lim_k (A\varphi_k, A\psi_k)_{L^2}$$

$$= \lim (\varphi_k, \psi_k)_{L^2}$$

$$= \lim_k (\varphi_k, \psi_k)_{L^2}$$

$$= (f, g)$$

$$\Rightarrow \quad (Af, Ag) = (f, g) \quad \forall f, g \in L^2$$

$$\Rightarrow \quad (f, A^*Ag) = (f, g) \quad \forall f, g$$

$$\Rightarrow \quad (f, A^*Ag - g) = 0 \quad \forall f, g$$

$$\Rightarrow \quad A^*Ag - g = 0 \quad \forall g$$

$$\left[\begin{array}{l} (x, z) = 0 \ \forall x \\ \Rightarrow (z, z) = 0 \\ \Rightarrow z = 0 \end{array}\right]$$

$$\Rightarrow \quad A^*Ag = g \quad \forall g$$

$$\Rightarrow \quad A^*A = \mathrm{Id}_{L^2 \to L^2}$$

In the same way $AA^* = \mathrm{Id}_{L^2 \to L^2}$

$$\Rightarrow \quad A^{-1} = A^* \qquad \blacksquare$$

Remark Let X Banach space and $D \subseteq X$ be a dense subspace (Ex.: $X = L^2$, $D = \mathcal{S}$). Let $T : D \to D$ be linear and

$$\|Tx\| \leq M \|x\| \quad \forall x \in D \qquad [\text{Ex. } T = \mathcal{F}]$$

Then there exists a unique $\tilde{T} \in \mathcal{L}(X)$ such that $\tilde{T} = T$ on D. In fact,

$$\tilde{T}x = \lim Tx$$

$$\tilde{T}x = \lim_k Tx_k$$

where $x \in X$ and $(x_k)_k \subset D$ with $x_k \longrightarrow x$ in X.