Lezione 34

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$\mathcal{F}: \mathcal{S} \longrightarrow \mathcal{S}$ bijective,

$$\boxed{(\mathcal{F}^{-1}\psi)(x) = \check{\psi}(x) = (2\pi)^{-n} \int e^{ix\xi}\, \psi(\xi)\, d\xi}$$

Tempered distributions

$$\mathcal{S}'(\mathbb{R}^n) = \{ T : \mathcal{S} \longrightarrow \mathbb{C} \mid T \text{ lin. \& cont.} \}$$

If $\varphi_k \longrightarrow \varphi$ in $\mathcal{S}$
then $T(\varphi_k) \longrightarrow T(\varphi)$ in $\mathbb{C}$

<u>Remark</u> $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ and

$$\phi_k \longrightarrow \phi \text{ in } \mathcal{D} \implies \phi_k \longrightarrow \phi \text{ in } \mathcal{S}.$$

Therefore $\mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$.

<u>Theorem</u> $T : \mathcal{S} \longrightarrow \mathbb{C}$ linear. Then

$$T \in \mathcal{S}' \iff \exists N \in \mathbb{N}_0 \;\exists C \geq 0 \;\forall \varphi \in \mathcal{S} :$$

$$|T(\varphi)| \leq C \, \|\varphi\|_{(N)}$$

<u>Basic concepts</u>:

- Regular distribution $T \in \mathcal{S}'$:

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$$\boxed{T(\varphi) = \int \underbrace{u(x)}\,\varphi(x)\,dx = T_u(\varphi)} \quad , \varphi \in \mathcal{S}$$

$$\exists N : \quad \frac{u(x)}{1+|x|^N} \in L^1(\mathbb{R}^n)$$

Example : • $L_p(\mathbb{R}^n) \subseteq \mathcal{S}'$ $\quad (1 \le p \le +\infty)$

• $u(x) = e^x \in L^1_{loc} \quad \Rightarrow \quad T_u \in \mathcal{D}'$ but $T_u \notin \mathcal{S}'$

- Differentiation as before,

$$\boxed{(\partial^\alpha T)(\varphi) = (-1)^{|\alpha|}\, T(\partial^\alpha \varphi)}$$

- Multiplication with functions :

$$\boxed{(aT)(\varphi) = T(a\varphi)}$$

where $a \in C^\infty(\mathbb{R}^n)$ is of tempered growth i.e.

$$\forall \alpha \; \exists N : \quad \sup_{x \in \mathbb{R}^n} |\partial^\alpha a(x)|\,(1+|x|)^{-N} < +\infty$$

[Then $\varphi \mapsto a\varphi : \mathcal{S} \to \mathcal{S}$ is continuous]

Example: • $a(x) = \sum_{|\alpha| \le N} a_\alpha x^\alpha$ (polynomial) ✓

• $a(x) = e^x$ not of tempered growth.

- Convolution : $\quad T \in \mathcal{S}', \; \varphi \in \mathcal{S} \quad \Rightarrow$

$$\boxed{(T * \varphi)(x) = T(\varphi(x-\bullet)) \quad , x \in \mathbb{R}^n}$$

Then $T * \varphi \in C^\infty$ is of tempered growth.

Fouriertransform of distributions:

$$T_{\hat{f}}(\varphi) = \int \hat{f}(x)\varphi(x)\,dx = \int f(x)\hat{\varphi}(x)\,dx$$

$$= T_f(\hat{\varphi}) \qquad \forall \varphi \in \mathcal{S}$$

<u>Definition</u> $T \in \mathcal{S}'$. Then

$$\boxed{\hat{T}(\varphi) = T(\hat{\varphi}) \quad , \varphi \in \mathcal{S}} \qquad [\hat{T} = \mathcal{F}T]$$

defines $\hat{T} \in \mathcal{S}'$ [because $\varphi \mapsto \hat{\varphi} : \mathcal{S} \to \mathcal{S}$ cont.]

Moreover, $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ bijective with

$$\boxed{(\mathcal{F}^{-1}T)(\varphi) = \check{T}(\varphi) = T(\check{\varphi}) \quad , \varphi \in \mathcal{S}}$$

<u>Example</u> $\delta = \delta$-distribution, $\delta(\varphi) = \varphi(0)$, $\delta \in \mathcal{S}'(\mathbb{R}^n)$.

$$\partial^\alpha \delta(\varphi) = (-1)^{|\alpha|}\delta(\partial^\alpha \varphi) = (-1)^{|\alpha|}(\partial^\alpha \varphi)(0)$$

$$\widehat{\partial^\alpha \delta}(\varphi) = \partial^\alpha \delta(\hat{\varphi}) = (-1)^{|\alpha|}\delta(\underbrace{\partial^\alpha \hat{\varphi}}_{= (-i)^{|\alpha|}\widehat{x^\alpha \varphi}})$$

$$= \delta(\widehat{x^\alpha \varphi}) \cdot i^{|\alpha|}$$

$$= \widehat{x^\alpha \varphi}(0) \cdot i^{|\alpha|}$$

$$= \widehat{x^\alpha \varphi}(0) \cdot i^{|\alpha|}$$

$$= \int e^{-ix0}\, x^\alpha \varphi(x)\, dx \cdot i^{|\alpha|}$$

$$= \int (ix)^\alpha \varphi(x)\, dx$$

$$= T_{(ix)^\alpha}(\varphi)$$

$$\Rightarrow \quad \boxed{\widehat{\partial^\alpha \delta} = T_{(ix)^\alpha}}$$

In part.: $\hat{\delta} = T_1$, $\hat{\delta}(\varphi) = \int \varphi(x)\, dx$

<u>Theorem</u> Let $f \in L^1(\mathbb{R}^n)$ with $\hat{f} \in L^1(\mathbb{R}^n)$. Then

$$\boxed{f(x) = (2\pi)^{-n} \int e^{ix\xi} \hat{f}(\xi)\, d\xi \qquad \text{a.e. } x \in \mathbb{R}^n.}$$

PROOF: $g(x) := (2\pi)^{-n} \int e^{ix\xi} \hat{f}(\xi)\, d\xi$, $g \in L^\infty$

$$T_f(\varphi) = (\mathcal{F}^{-1} \widehat{T_f})(\varphi) = T_{\hat{f}}(\check{\varphi})$$

$$= \int \hat{f}(\xi) \left((2\pi)^{-n} \int e^{ix\xi} \varphi(x)\, dx \right) d\xi$$

$$\overset{\text{Fubini}}{=} \int \left((2\pi)^{-n} \int e^{ix\xi} \hat{f}(\xi)\, d\xi \right) \varphi(x)\, dx$$

$$= \int g(x) \varphi(x)\, dx$$

$$= T_g(\varphi) \qquad \forall \varphi \in \mathcal{S}$$

We know: $T_f = T_g \iff f = g$ a.e. in $\mathbb{R}^n$

$\blacksquare$

Distributions and partial diff. eq

Lemma $T \in \mathcal{S}'$

$$\boxed{\widehat{\partial^\alpha T} = i^{|\alpha|}\, \xi^\alpha\, \hat{T}\,, \quad (-i)^{|\alpha|}\, \widehat{x^\alpha T} = \partial^\alpha \hat{T}}$$

(Homework!)

Let $A = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha$ be diff. operator with constant coefficients ($a_\alpha \in \mathbb{C}$). Define

$$\boxed{a(\xi) = \sum_{|\alpha| \le m} a_\alpha (i\xi)^\alpha = \sigma_A(\xi)} \qquad \text{"symbol of A"}$$

If $T \in \mathcal{S}'$ then

$$\widehat{AT} = \mathcal{F}\Big(\sum_{|\alpha| \le m} a_\alpha \partial^\alpha T\Big) = \sum_{|\alpha| \le m} a_\alpha \widehat{\partial^\alpha T}$$

$$= \sum_{|\alpha| \le m} a_\alpha (i\xi)^\alpha \hat{T}$$

$$= a \hat{T}$$

Suppose now that <u>a has no zeros</u> ($a(\xi) \neq 0 \ \forall \xi \in \mathbb{R}^n$).

Suppose now [illegible]

Given $S \in \mathcal{S}'$ look for $T \in \mathcal{S}'$ with

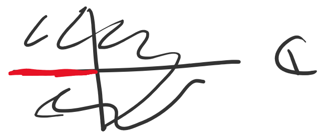
$$AT = S$$

Now

$$\begin{aligned} AT = S &\iff \widehat{AT} = \hat{S} \\ &\iff a\hat{T} = \hat{S} \\ &\iff \hat{T} = \frac{1}{a}\hat{S} \\ &\iff \boxed{T = \mathcal{F}^{-1}\left(\frac{1}{a}\hat{S}\right)} \end{aligned}$$

$\Rightarrow$ $\exists$ unique solution $T \in \mathcal{S}'$ with "solution formula"

<u>Example</u> $\Delta = \partial_{x_1}^2 + \ldots + \partial_{x_n}^2$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$



$$A = \lambda - \Delta$$

$$\begin{aligned} a(\xi) &= \lambda - \left(-\xi_1^2 - \ldots - \xi_n^2\right) \\ &= \lambda + \left(\xi_1^2 + \ldots + \xi_n^2\right) = \lambda + |\xi|^2 \end{aligned}$$

$$a(\xi) \neq 0 \quad \forall \xi \iff \lambda \notin (-\infty, 0]$$

$\Rightarrow$ $\lambda - \Delta : \mathcal{S}' \longrightarrow \mathcal{S}'$ is bijective

$$\boxed{(\lambda - \Delta)^{-1} S = \mathcal{F}^{-1}\left(\frac{1}{\lambda + |\xi|^2}\hat{S}\right)}$$

<u>Remark</u> For $s \in \mathbb{R}$ define

$$H^s(\mathbb{R}^n) = \{ T \in \mathcal{S}' \mid \underbrace{(1+|\xi|^2)^{s/2} \hat{T}}_{\substack{\text{is a regular} \\ \text{distr. with} \\ \text{density in } L^2}} \in L^2(\mathbb{R}^n) \}$$

"Sobolev spaces". If $k \in \mathbb{N}_0$ then

- $H^k(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$
- $f \in H^k(\mathbb{R}^n) \iff \partial^\alpha T_f \in L^2 \quad \forall |\alpha| \leq k$
- $A = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n)$
- $(\lambda - \Delta) : H^s(\mathbb{R}^n) \longrightarrow H^{s-2}(\mathbb{R}^n)$ is bijective

  for $\lambda \notin (-\infty, 0]$.

<u>Heisenberg uncertainty principle</u>

<u>Theorem</u> Let $\varphi \in \mathcal{S}(\mathbb{R})$ and $x_0, \xi_0 \in \mathbb{R}$. Then

$$\boxed{\|\varphi\|_{L^2}^2 \leq \sqrt{\frac{2}{\pi}} \, \|(x-x_0)\varphi\|_{L^2} \, \|(\xi-\xi_0)\hat{\varphi}\|_{L^2}}$$

"inverse proportional"

PROOF: Step 1: $x_0 = \xi_0 = 0$

$$\|\varphi\|_{L^2}^2 = \int \varphi(x)\overline{\varphi(x)}\,dx$$

$$= - \int x\,(\varphi\overline{\varphi})'(x)\,dx$$

$$= - \int x\,\left(\varphi'\overline{\varphi} + \varphi\overline{\varphi'}\right)(x)\,dx$$

$$= -2 \int x \, \mathrm{Re}\,(\varphi'\overline{\varphi})(x)\,dx \qquad \left(\mathrm{Re}\,z = \frac{z+\overline{z}}{2}\right)$$

$$\Rightarrow \|\varphi\|_{L^2}^2 \leq 2 \int |x|\,|\mathrm{Re}\,(\varphi'\overline{\varphi})(x)|\,dx \qquad \left(|\mathrm{Re}\,z| \leq |z|\right)$$

$$\leq 2 \int |x\varphi(x)|\,|\varphi'(x)|\,dx$$

$$\leq 2\,\|x\varphi(x)\|_{L^2}\,\underbrace{\|\varphi'\|_{L^2}}_{= \frac{1}{\sqrt{2\pi}}\|\widehat{\varphi'}\|_{L^2} = \frac{1}{\sqrt{2\pi}}\|\xi\hat{\varphi}\|_{L^2}} \qquad \left(|(f,g)| \leq \|f\|\|g\|\right)$$

Cauchy-Schwartz

$$\leq \sqrt{\frac{2}{\pi}}\,\|x\varphi\|_{L^2}\,\|\xi\hat{\varphi}\|_{L^2}$$

Step 2: $\psi(x) := e^{-ix\xi_0}\varphi(x+x_0) \Rightarrow$

- $\|\psi\|_{L^2} = \|\varphi\|_{L^2}$
- $\|x\psi\|_{L^2} = \|(x-x_0)\varphi\|_{L^2}$
- $\|\xi\hat{\psi}\|_{L^2} = \|(\xi-\xi_0)\hat{\varphi}\|_{L^2}$

$$\|\psi\|_{L^2} \le \sqrt{\frac{2}{\pi}}\, \|x\psi\|_{L^2}\, \|\xi\hat{\psi}\|_{L^2}$$

$\Rightarrow$ Formula for φ. ∎

<u>Theorem (Amrein-Berthier)</u> Let $E, F \subseteq \mathbb{R}$ of finite (Lebesgue-) measure. Then there exists a $C \geq 0$ s.t.

$$\boxed{\|f\|_{L^2(\mathbb{R}^n)} \le C\left(\|f\|_{L^2(\mathbb{R}^n \setminus E)} + \|\hat{f}\|_{L^2(\mathbb{R}^n \setminus F)}\right)}$$

<u>Application</u> Suppose $\phi \in \mathcal{D}(\mathbb{R})$ and $\hat{\phi} \in \mathcal{D}(\mathbb{R})$.

Take $E = \operatorname{supp} \phi$, $F = \operatorname{supp} \hat{\phi}$

$$\|\phi\|_{L^2(\mathbb{R})} \le C\,(0 + 0) = 0$$

$\Rightarrow \phi = 0$