The Laplace transform

Is a tool to solve ode like

$$\begin{cases} y''(x) + a_1 y'(x) + a_0 y(x) = b(x) , \quad x>0 \\ y(0) = y_0 , \quad y'(0) = y_1 \end{cases}$$

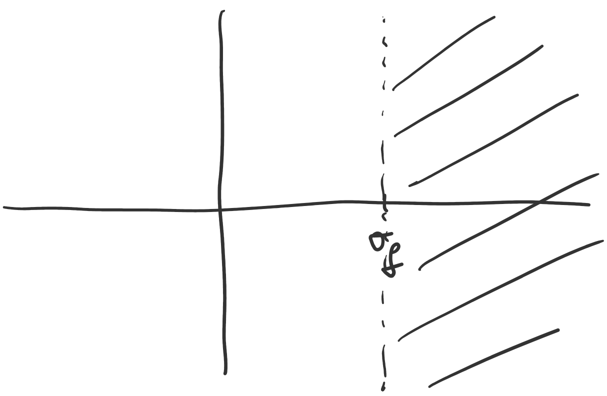
Definition $f : (0,+\infty) \to \mathbb{R}$ is called $\mathcal{L}$-transformable if

$$\boxed{\exists \sigma \in \mathbb{R} : \quad x \mapsto e^{-\sigma x} f(x) \in L^1((0,+\infty))} \qquad (*)$$

$$\sigma_f := \inf \{ \sigma \in \mathbb{R} \mid \sigma \text{ satisfies } (*) \}$$

Define

$$\boxed{(\mathcal{L} f)(s) = \left(\mathcal{L} f(x)\right)(s) = \int_0^{+\infty} e^{-sx} f(x)\, dx , \quad s \in \mathbb{C} , \ \operatorname{Re} s > \sigma_f}$$



Nota • $e^z = \sum_{n=0}^{+\infty} \frac{1}{n!} z^n$, $|e^z| = e^{\operatorname{Re} z}$

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• $|e^{-sx} f(x)| = e^{-(\operatorname{Re} s)x} |f(x)| \in L^1((0,+\infty))$, $\operatorname{Re} s > \sigma_f$

Example $f(x) = e^{ax}$, $a \in \mathbb{C}$.

$$(\mathcal{L} e^{ax})(s) = \int_0^{+\infty} e^{(a-s)x}\, dx$$

$$\overset{a \neq s}{=} \left. \frac{e^{(a-s)x}}{a-s} \right|_0^{+\infty}$$

$$= \lim_{b \to +\infty} \left. \frac{e^{(a-s)x}}{a-s} \right|_{x=0}^{x=b}$$

$$= \frac{1}{a-s} \qquad (\operatorname{Re} s > \operatorname{Re} a =: \sigma_f)$$

$$\left[\left| e^{(a-s)x} \right| = e^{(\operatorname{Re} a - \operatorname{Re} s)x} \xrightarrow{x \to +\infty} 0 \right]$$

$$\boxed{(\mathcal{L} e^{ax})(s) = \frac{1}{a-s} , \ \operatorname{Re} s > \operatorname{Re} a}$$

Example Let $\omega \in \mathbb{R}$.

$$\cos(\omega x) = \frac{e^{i\omega x} + e^{-i\omega x}}{2}$$

$$\sin(\omega x) = \frac{e^{i\omega x} - e^{-i\omega x}}{2i}$$

$$\left[\begin{array}{l} e^{i\theta} = \cos\theta + i \sin\theta \\ \operatorname{Re} z = \frac{z + \bar{z}}{2} \\ \operatorname{Im} z = \frac{z - \bar{z}}{2i} \end{array} \right]$$

Use previous example with $a = \pm i\omega$

$$\boxed{\begin{aligned}(\mathcal{L}\cos(\omega x))(s) &= \frac{s}{s^2+\omega^2}\\ (\mathcal{L}\sin(\omega x))(s) &= \frac{\omega}{s^2+\omega^2}\end{aligned}} \quad , \operatorname{Re} s > 0$$

<u>Example</u> $k \in \mathbb{N}$

$$(\mathcal{L}x^k)(s) = \int_0^{+\infty} e^{-sx} x^k \, dx \qquad (\operatorname{Re} s > 0)$$

$$= \int_0^{+\infty} \frac{1}{-s} \left(e^{-sx}\right)' x^k \, dx$$

$$= \underbrace{-\frac{1}{s} e^{-sx} x^k \Big|_{x=0}^{x=+\infty}}_{=0} + \frac{k}{s} \int_0^{+\infty} e^{-sx} x^{k-1} \, dx$$

$$(\mathcal{L}x^k)(s) = \frac{k}{s} (\mathcal{L} \, x^{k-1})(s)$$

$$= \frac{k}{s} \frac{k-1}{s} (\mathcal{L} \, x^{k-2})(s)$$

$$= \dots = \frac{k!}{s^k} (\mathcal{L} 1)(s) \qquad [1 = x^0 = e^{0x}]$$

$$\boxed{(\mathcal{L}x^k)(s) = \frac{k!}{s^{k+1}}} \quad , \operatorname{Re} s > 0 .$$

<u>Lemma</u> Let f, g $\mathcal{L}$-transformable and $a, b \in \mathbb{R}$.

a) $\mathcal{L}(af + bg)(s) = a(\mathcal{L}f)(s) + b(\mathcal{L}g)(s)$, $\operatorname{Re} s > \max\{\sigma_f, \sigma_g\}$

b) $\mathcal{L}(e^{ax} f(x))(s) = (\mathcal{L}f)(s-a) \quad , \operatorname{Re} s > \sigma_f + \operatorname{Re} a$

$$\left[\int_0^{+\infty} e^{-sx} e^{ax} f(x)\, dx = \int_0^{+\infty} e^{-(s-a)x} f(x)\, dx \right]$$

c) $\mathcal{L}(x^n f(x))(s) = (-1)^n \frac{d^n}{ds^n} (\mathcal{L}f)(s)$

$$\left[\frac{d^n}{ds^n} \int_0^{+\infty} e^{-sx} f(x)\, dx = (-x)^n \int_0^{+\infty} e^{-sx} f(x)\, dx \right]$$

The inverse Laplace transform

$s = \sigma + i\tau \in \mathbb{C}$

$$(\mathcal{L}f)(\sigma + i\tau) = \int_0^{+\infty} e^{-(\sigma+i\tau)x} f(x)\, dx$$

$$= \int_0^{+\infty} e^{-i\tau x} \left(e^{-\sigma x} f(x) \right) dx$$

$$= \int_{-\infty}^{+\infty} e^{-i\tau x} \left(e^{-\sigma x} \tilde{f}(x) \right) dx$$

$$\tilde{f}(x) = \begin{cases} f(x) & , x > 0 \\ 0 & , x < 0 \end{cases}$$

$$\boxed{(\mathcal{L}f)(\sigma + i\tau) = \mathcal{F}\Big(\underbrace{e^{-\sigma x} \tilde{f}}_{\in L^1(\mathbb{R})} \Big)(\tau)} \qquad (\sigma > \sigma_f)$$

Therefore

- The Laplace transform is injective

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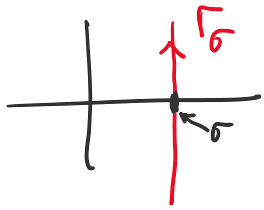
  (if $\mathcal{L}f$ and $\mathcal{L}g$ coincide on some vertical line $\{\sigma + i\tau \mid \tau \in \mathbb{R}\}$, $\sigma > \max(\sigma_f, \sigma_g)$, then $f = g$ a.e.)

- If $\tau \mapsto (\mathcal{L}f)(\sigma + i\tau) \in L^1(\mathbb{R})$ one recovers f:

$$f(x) = e^{\sigma x} \left(\mathcal{F}^{-1}_{\tau \to x} (\mathcal{L}f)(\sigma + i\tau) \right)(x)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(\sigma + i\tau)x} (\mathcal{L}f)(\sigma + i\tau)\, d\tau$$

$$\left[= \frac{1}{2\pi i} \int_{\Gamma_\sigma} e^{zx} (\mathcal{L}f)(z)\, dz \right.$$



where Γ_σ is curve with parametrization $\gamma(\tau) = \sigma + i\tau$

- $\int_\Gamma g(z)\, dz = \int_a^b g(\gamma(\tau)) \gamma'(\tau)\, d\tau$
  where $\gamma : [a,b] \to \mathbb{C}$ param. of Γ $\Big]$

$$= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{sx} (\mathcal{L}f)(s)\, ds$$

<u>Definition</u> For suitable F holomorphic for $\mathrm{Re}\, s \geq \sigma$,

$$\boxed{(\mathcal{L}^{-1} F)(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{sx} F(s)\, ds}$$

Example

$$\mathcal{L}(x^n e^{ax})(s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L}(e^{ax})(s)$$

$$= (-1)^n \frac{d^n}{ds^n} (s-a)^{-1}$$

$$= n!\ (s-a)^{-(n+1)} = \frac{n!}{(s-a)^{n+1}}$$

$$\Rightarrow \boxed{\left(\mathcal{L}^{-1} \frac{1}{(s-a)^n}\right)(x) = \frac{x^{n-1}}{(n-1)!} e^{ax}}$$

Laplace transform and convolution

$f, g : (0, +\infty) \to \mathbb{C}$ "suitable"

$$\boxed{(f * g)(x) := \int_0^x f(x-y) g(y)\, dy \quad , x > 0}$$

Theorem f, g $\mathcal{L}$-transformable

$$\mathcal{L}(f * g) = (\mathcal{L}f) \cdot (\mathcal{L}g)$$

$$\mathcal{L}^{-1}(FG) = (\mathcal{L}^{-1}F) * (\mathcal{L}^{-1}G)$$

Application to ode

$\Big[$ $A = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha$

$$Au = f \iff \mathcal{F}(Au) = \hat{f} \iff a(\xi)\hat{u}(\xi) = \hat{f}(\xi) \quad \forall \xi$$

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$$\iff u = \mathcal{F}^{-1}\left(\frac{1}{a}\,\hat{f}\right)$$

<u>Theorem</u> Let $f \in C^N([0,+\infty))$ such that $f, f', f'', \dots, f^{(N)}$ are $\mathcal{L}$-transformable. Then

$$\boxed{(\mathcal{L} f^{(j)})(s) = s^j (\mathcal{L}f)(s) - \sum_{k=1}^{j} f^{(k-1)}(0)\, s^{j-k}}$$

$\operatorname{Re} s > \sigma_f$

PROOF:

$$(\mathcal{L}f')(s) = \int_0^{+\infty} e^{-sx} f'(x)\,dx$$

$$= e^{-sx} f(x)\Big|_{x=0}^{x=+\infty} + s\int_0^{+\infty} e^{-sx} f(x)\,dx$$

$$= s(\mathcal{L}f)(s) - f(0)$$

$$(\mathcal{L}f'')(s) = s(\mathcal{L}f')(s) - f'(0)$$

$$= s\Big(s(\mathcal{L}f)(s) - f(0)\Big) - f'(0)$$

$$= s^2(\mathcal{L}f)(s) - \Big(f'(0) + f(0)s\Big)$$

$$(\mathcal{L}f''')(s) = s(\mathcal{L}f'')(s) - f''(0)$$

$$= \dots$$

$\blacksquare$

<u>Application</u>

Applications

$$\left.\begin{array}{r} y''(x) + a_1 y'(x) + a_0 y(x) = b(x) \\ y(0) = y_0 \ , \ y'(0) = y_1 \end{array}\right\}$$

$a_0, a_1, y_0, y_1, b(x)$ are given, $y(x)$ unknown

Apply the Laplace transform:

$$s^2 (\mathcal{L}y)(s) - \left(y'(0) + y(0)s\right) +$$

$$+ a_1 \left(s(\mathcal{L}y)(s) - y(0)\right) + a_0 (\mathcal{L}y)(s) = (\mathcal{L}b)(s)$$

$$\underbrace{\left(s^2 + a_1 s + a_0\right)}_{=:P(s)} (\mathcal{L}y)(s) - \underbrace{\left(y_1 + y_0 s + a_1 y_0\right)}_{=:Q(s)} = (\mathcal{L}b)(s)$$

characteristic polynomial of the equation

$$\boxed{(\mathcal{L}y)(s) = \frac{Q(s) + (\mathcal{L}b)(s)}{P(s)} = \frac{Q(s)}{P(s)} + \frac{(\mathcal{L}b)(s)}{P(s)}}$$

Apply inverse Laplace transform:

$$y(x) = \left(\mathcal{L}^{-1} \frac{Q + \mathcal{L}b}{P}\right)(x)$$

$$\mathcal{L}^{-1}\left(\frac{1}{P}\mathcal{L}b\right)$$

$$\boxed{y(x) = \left(\mathcal{L}^{-1}\frac{Q}{P}\right)(x) + \left(\mathcal{L}^{-1}\frac{1}{P}\right) * b}$$