Laplace transform $\mathcal{L}$

$$(\mathcal{L}e^{ax})(s) = \frac{1}{s-a} \qquad (a \in \mathbb{C})$$

$$(\mathcal{L}\cos(\omega x))(s) = \frac{s}{s^2+\omega^2}$$

$$(\omega \in \mathbb{R})$$

$$(\mathcal{L}\sin(\omega x))(s) = \frac{\omega}{s^2+\omega^2}$$

$$(\mathcal{L}(e^{ax}f(x)))(s) = (\mathcal{L}f)(s-a) \qquad (a \in \mathbb{C})$$

$$\left(\mathcal{L}^{-1}\frac{1}{(s-a)^n}\right)(x) = \frac{x^{n-1}}{(n-1)!}e^{ax}$$

$$(\mathcal{L}f^{(j)})(s) = s^j(\mathcal{L}f)(s) - \sum_{k=1}^{j} f^{(k-1)}(0)\, s^{j-k}$$

$$\left.\begin{aligned} &y''(x) + a_1 y'(x) + a_2 y(x) = b(x) \qquad , x>0 \\ &y(0)=y_0 \, , \; y'(0)=y_1 \end{aligned}\right\}$$

$$\overset{\mathcal{L}}{\Longrightarrow} \quad \underbrace{(s^2+a_1s+a_0)}_{=:P(s)}(\mathcal{L}y)(s) - \underbrace{(y_0s+y_1+a_1y_0)}_{=:Q(s)} = (\mathcal{L}b)(s)$$

$$\overset{\mathcal{L}^{-1}}{\Longrightarrow} \quad u(x) = \left(\mathcal{L}^{-1}\frac{Q}{P}\right)(x) + b * \left(\mathcal{L}^{-1}\frac{1}{P}\right)(x)$$

$$\overset{\mathcal{L}^{-1}}{\Longrightarrow} \quad \boxed{y(x) = \left(\mathcal{L}^{-1}\frac{Q}{P}\right)(x) + b * \left(\mathcal{L}^{-1}\frac{1}{P}\right)(x)}$$

$$= \left(\mathcal{L}^{-1}\frac{Q + \mathcal{L}b}{P}\right)(x)$$

Partial fraction decomposition

Let $P(s) = (s-\lambda_0)(s-\lambda_1)$ where $\lambda_0 \neq \lambda_1$, $\lambda_j \in \mathbb{C}$.

$$\boxed{\frac{Q(s)}{P(s)} \overset{!}{=} \frac{a}{s-\lambda_0} + \frac{b}{s-\lambda_1}}$$

$$= \frac{(a+b)s - a\lambda_1 - b\lambda_0}{(s-\lambda_0)(s-\lambda_1)}$$

with

$$\boxed{\begin{aligned} a &= \left.\frac{Q(s)}{s-\lambda_1}\right|_{s=\lambda_0} = \frac{Q(\lambda_0)}{\lambda_0-\lambda_1} \\ b &= \left.\frac{Q(s)}{s-\lambda_0}\right|_{s=\lambda_1} = \frac{Q(\lambda_1)}{\lambda_1-\lambda_0} \end{aligned}}$$

or, equivalently,

$$\boxed{a + b = y_0, \quad -a\lambda_1 - b\lambda_0 = y_1 + a_1 y_0}$$

Exercise Let $P(s) = (s-\lambda)^2$, $\lambda \in \mathbb{C}$.

$$\frac{Q(s)}{P(s)} = \frac{a}{s-\lambda} + \frac{b}{(s-\lambda)^2}$$

Find a and b.

Example

$$y''(x) - \overset{a_1=-2}{2}y'(x) - \overset{a_0=-8}{8}y(x) = 0 \leftarrow b(x)=0$$

$$\underset{y_0=1}{y(0)=1}, \quad \underset{y_1=2}{y'(0)=2}$$

Characteristic polynomial:

$$P(s) = s^2 - 2s - 8 = (s+2)(s-4)$$

and

$$Q(s) = s.$$

Then

$$\frac{Q(s)}{P(s)} = \frac{a}{s+2} + \frac{b}{s-4}$$

with

$$a = \left.\frac{s}{s-4}\right|_{s=-2} = \frac{1}{3}$$

$$b = \left.\frac{s}{s+2}\right|_{s=4} = \frac{2}{3}$$

$$\Rightarrow y(x) = \left(\mathcal{L}^{-1}\frac{P}{Q}\right)(x)$$

$$= \frac{1}{3}\left(\mathcal{L}^{-1}\frac{1}{s+2}\right)(x) + \frac{2}{3}\left(\mathcal{L}^{-1}\frac{1}{s-4}\right)(x)$$

$$\boxed{y(x) = \frac{1}{3} e^{-2x} + \frac{2}{3} e^{4x}}$$

<u>Example</u>

$$y''(x) - 2y'(x) + 2y(x) = b(x)$$

$$y(0)=2, \quad y'(0)=3$$

Then

$$P(s) = s^2 - 2s + 2$$

$$= \big(s - (1+i)\big)\big(s-(1-i)\big)$$

$\Big[$ $P(s) = s^2 + a_1 s + a_0$ with $a_0, a_1 \in \mathbb{R}$

λ zero of P $\Rightarrow$ $\bar{\lambda}$ zero of P

$P(\bar{\lambda}) = \bar{\lambda}^2 + a_1\bar{\lambda} + a_0 = \overline{\lambda^2 + a_1\lambda + a_0} = \bar{0} = 0$ $\Big]$

$$Q(s) = 2s - 1$$

$$\frac{Q(s)}{P(s)} = \frac{a}{s-(1+i)} + \frac{b}{s-(1-i)}$$

$$a = \left.\frac{2s-1}{s-(1-i)}\right|_{s=1+i} = \frac{1+2i}{2i} = 1 + \frac{1}{2i} = 1 - \frac{i}{2}$$

$$b = \left.\frac{2s-1}{s-(1+i)}\right|_{s=1-i} = \frac{1-2i}{-2i} = 1+\frac{i}{2}$$

$$\Rightarrow \left(\mathcal{L}^{-1}\frac{Q}{P}\right)(x) = \left(1-\frac{i}{2}\right)\left(\mathcal{L}^{-1}\frac{1}{s-(1+i)}\right)(x) +$$

$$+\left(1+\frac{i}{2}\right)\left(\mathcal{L}^{-1}\frac{1}{s-(1-i)}\right)(x)$$

$$= \left(1-\frac{i}{2}\right) e^{(1+i)x} + \left(1+\frac{i}{2}\right) e^{(1-i)x}$$

$$= e^{x}\left(\left(1-\frac{i}{2}\right)e^{ix} + \left(1+\frac{i}{2}\right)e^{-ix}\right)$$

$$= e^{x}\left(e^{ix}+e^{-ix} + \frac{i}{2}\left(e^{-ix}-e^{ix}\right)\right)$$

$$= e^{x}\left(2\cos x + \sin x\right) \qquad \left[e^{\pm i\theta} = \cos\theta \pm i\sin\theta\right]$$

$$= \hat{y}(x)$$

$\hat{y}(x)$ is the solution to

$$y''(x) - 2y'(x) + 2y(x) = 0$$
$$y(0) = 2, \; y'(0) = 3$$

Now

$$\frac{1}{P(s)} = \frac{1}{2i}\left(\frac{1}{s-(1+i)} - \frac{1}{s-(1-i)}\right)$$

$$\left(\mathcal{L}^{-1}\frac{1}{p}\right)(x) = \frac{1}{2i}\left(e^{(1+i)x} - e^{(1-i)x}\right)$$

$$= e^{x}\sin x$$

$$\boxed{(f*g)(x) = \int_0^x f(y)g(x-y)\,dy}$$

Thus

$$\left(b*\mathcal{L}^{-1}\frac{1}{p}\right)(x) = e^{x}\int_0^x b(y)\,e^{-y}\sin(x-y)\,dy$$

and

$$y(x) = \hat{y}(x) + b*\left(\mathcal{L}^{-1}\frac{1}{p}\right)(x)$$

$$\boxed{y(x) = e^{x}\underset{(2\cos x+\sin x)}{\uparrow} + e^{x}\int_0^x b(y)\,e^{-y}\sin(x-y)\,dy}$$

Example

$$y''(x) + 2y'(x) + 5y(x) = e^{-x}\sin x$$

$$y(0)=0,\quad y'(0)=1$$

$P(s) = s^2+2s+5$ has no real zeros.

Applying $\mathcal{L}$:

$$(s^2+2s+5)(\mathcal{L}y)(s) - 1 = \frac{1}{(s+1)^2+1}$$

$$(\mathcal{L}y)(s) = \frac{s^2+2s+3}{}$$

$$(\mathcal{L}y)(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

(Real) partial fraction decomposition:

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$= \frac{\ldots\ldots}{(s^2+2s+2)(s^2+2s+5)}$$

leads to

$$\begin{aligned} A + C &= 0 \\ 2A + B + 2C + D &= 1 \\ 5A + 2B + 2C + 2D &= 2 \\ 5B + 2D &= 3 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 \\ 5 & 2 & 2 & 2 & 2 \\ 0 & 5 & 0 & 2 & 3 \end{array}\right) \longrightarrow \text{Gauss-Algorithm}$$

$$\Rightarrow A = C = 0, \quad B = \frac{1}{3}, \quad D = \frac{2}{3}$$

$$y(x) = \frac{1}{3}\left(\mathcal{L}^{-1}\frac{1}{s^2+2s+2}\right)(x) + \frac{2}{3}\left(\mathcal{L}^{-1}\frac{1}{s^2+2s+5}\right)(x)$$

"Completion of squares"

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$$s^2 + as + b = \left(s + \frac{a}{2}\right)^2 - \frac{a^2}{4} + b$$

$$y(x) = \frac{1}{3}\left(\mathcal{L}^{-1} \frac{1}{(s+1)^2+1}\right)(x) + \frac{1}{3}\left(\mathcal{L}^{-1} \frac{2}{(s+1)^2+4}\right)(x)$$

$$\boxed{y(x) = \frac{1}{3} e^{-x}\left(\sin x + \sin(2x)\right)}$$

<u>Example</u>

$$y_1'(x) + 2y_2'(x) - 3y_2(x) = 12e^x$$

$$y_1'(x) - y_2'(x) - y_1(x) = 6$$

$$y_1(0) = y_2(0) = 0$$

$$\mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

$$\mathbf{A}\mathbf{y}'(x) + \mathbf{B}\mathbf{y}(x) = \mathbf{b}(x)$$

$$\mathbf{y}(0) = \mathbf{0}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{b}(x) = \begin{pmatrix} 12e^x \\ 6 \end{pmatrix}$$

Apply $\mathcal{L}$:

$$s(\mathcal{L}y_1)(s) + (2s-3)(\mathcal{L}y_2)(s) = 12\,\frac{1}{s-1}$$

$$(s-6)(\mathcal{L}y_1)(s) - s(\mathcal{L}y_2)(s) = 6\,\frac{1}{s}$$

$$\underbrace{\begin{pmatrix} s & 2s-3 \\ s-6 & -s \end{pmatrix}}_{=:A(s)} \begin{pmatrix} (\mathcal{L}y_1)(s) \\ (\mathcal{L}y_2)(s) \end{pmatrix} = 6 \begin{pmatrix} \frac{2}{s-1} \\ \frac{1}{s} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A(s)^{-1} = -\frac{1}{3(s-2)(s-3)} \begin{pmatrix} -s & 3-2s \\ 6-s & s \end{pmatrix}$$

We find

$$(\mathcal{L}y_1)(s) = \frac{1}{2}\,\frac{1}{s-1} - 3\,\frac{1}{s-2} + \frac{7}{2}\,\frac{1}{s-3} - \frac{1}{s}$$

$$(\mathcal{L}y_2)(s) = 6\,\frac{1}{s-2} - \frac{7}{2}\,\frac{1}{s-3} - \frac{5}{2}\,\frac{1}{s-1}$$

Applying $\mathcal{L}^{-1}$:

$$\boxed{\begin{aligned} y_1(x) &= \frac{1}{2}e^{x} - 3e^{2x} + \frac{7}{2}e^{3x} - 1 \\ y_2(x) &= 6e^{2x} - \frac{7}{2}e^{3x} - \frac{5}{2}e^{x} \end{aligned}}$$

$$\boxed{\begin{aligned} y_1(x) &= \frac{1}{2}e^{x} - 3e^{2x} + \frac{7}{2}e^{3x} - 1 \\ y_2(x) &= 6e^{2x} - \frac{7}{2}e^{3x} - \frac{5}{2}e^{x} \end{aligned}}$$