# Analysis (SDS – UNITO, 23/24)
# Week 1: normed and Banach spaces

S. Ivan Trapasso

**Exercise 1 (The maximum of two norms)**
Let $X$ be a normed space and $\|\cdot\|_\alpha, \|\cdot\|_\beta$ be norms on $X$.

Prove that $\|\cdot\|_M := \max\{\|\cdot\|_\alpha, \|\cdot\|_\beta\}$ is a norm on $X$.

**Exercise 2 (The minimum of two norms)**
Given $\lambda > 0$, consider the following norms for $f \in C[0,1]$:

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|, \qquad \|f\|_{1,\lambda} := \lambda \int_0^1 |f(x)|dx.$$

Prove that constraints on $\lambda$ must be set in order for $\|\cdot\|_m := \min\{\|\cdot\|_\infty, \|\cdot\|_{1,\lambda}\}$ to be a norm on $C[0,1]$.

**Exercise 3**
We prove below that the set $P$ of polynomials is not open in $C[-1,1]$. Fill the details.

(a) Assume by contradiction that $P$ is open. Consider $f(x) = x$. Since $P$ is open, there exists $R > 0$ such that $\ldots \subseteq P$.

(b) Consider now $h_R(x) := x + R\dfrac{x}{2(1+|x|)}$. We have

$$\|h_R - f\|_\infty \le \ldots,$$

hence $h_R \in \ldots$ – in particular, $h_R \in P$.

(c) We obtained above that $h_R$ is a polynomial, but...

Hence a contradiction.

**Exercise 4 (Topological properties of linear subspaces)**
Let $X$ be a normed space and $S \subseteq X$ a linear subspace.

(a) Prove that if $S$ is contained in a ball then $S = \{0\}$.

(b) Prove that if $S$ contains a ball then $S = X$.

**Exercise 5 (The space of convergent series)**
Consider the set

$$S = \Big\{a = (a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} : \sum_{n\in\mathbb{N}} a_n < \infty\Big\},$$

and for $a = (a_n)_{n\in\mathbb{N}} \in S$ set $\|a\|_S := \sup_{n\in\mathbb{N}} \left|\sum_{k=0}^{n} a_k\right|$.

Prove that $(S, \|\cdot\|_S)$ is a Banach space.

**Exercise 6 (On some spaces of polynomials)**
Let $P_k$ be the space of real polynomials of degree at most $k \in \mathbb{N}$.

(a) For $p \in P_k$ set $\|p\| = \sum_{j=0}^{k} |p(j)|$. Is $(P_k, \|\cdot\|)$ a Banach space?

(b) Prove that for any $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$|p(0)| \leq C_k \int_0^1 |p(x)|dx, \quad p \in P_k.$$

*Hint.* There is a short and nice argument that smartly exploits equivalence of norms.

**Exercise 7 (Interplay between continuity and norms)**
Consider the functional

$$T \colon C[0,1] \to \mathbb{R}, \qquad T(f) = f(1).$$

Prove that $T$ is continuous if $C[0,1]$ is endowed with the standard $\|\cdot\|_\infty$ norm, but continuity fails under any norm $\|\cdot\|_{L^p}$ with $1 \leq p < \infty$.

**Exercise 8**
A necessary condition for convergence Let $X$ be a normed space.

Show that if $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} \|x_n\| = \|x\|$.

What about the converse?

**Exercise 9 ($\star$)**
Let $X$ be a normed space, and fix $x, y \in X$ and $\lambda \in \mathbb{R}$. Show that

$$\lim_{n\to\infty} (\|(n+\lambda)x + y\| - \|nx + y\|) = \lambda\|x\|.$$

**Exercise 10 ($\star$ The space of Lipschitz continuous functions)**
Recall that $f \colon [0,1] \to \mathbb{R}$ is said to be Lipschitz continuous if there exists $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in [0,1]$.

Let $L$ be the set of Lipschitz continuous functions $[0,1] \to \mathbb{R}$. Is $L$ a closed subspace of $C[0,1]$?

*Hint.* Recall Weierstrass' approximation theorem.