Analysis (SDS – UNITO, 23/24)
Week 2: more on normed and Banach spaces

S. Ivan Trapasso

Exercise 1 *(Closed balls and closure of the open balls)*

Let (X, d) be a metric space. Prove that the following conditions are equivalent.

- For any $x \in X$ and $r > 0$, the closure $\overline{B(x,r)}$ of the open ball $B(x,r) = \{y \in X : d(y,x) < r\}$ coincides with the closed ball $\bar{B}(x,r) = \{y \in X : d(y,x) \le r\}$.
- For any $x, y \in X$ with $x \neq y$ and $\varepsilon > 0$, there exists $z \in X$ such that $d(z,y) < \varepsilon$ and $d(x,z) < d(x,y)$.

Exercise 2 *(Minkowski's gauge functional)*

Let $(X, \|\cdot\|)$ be a normed real vector space. Let $C \subset X$ be an open convex[1] set with $0 \in C$. The Minkowski gauge functional associated with C is defined as follows: for any $x \in C$ set

$$p_C(x) \coloneqq \inf\{t > 0 : t^{-1}x \in C\}.$$

Prove the following properties:

(a) $p_C(x)$ is well defined, that is: for each $x \in C$ we have $\{t > 0 : t^{-1}x \in C\} \neq \emptyset$.

(b) p_C is positively homogeneous, that is: $p_C(\alpha x) = \alpha p_C(x)$ for all $x \in C$ and $\alpha > 0$.

(c) p_C is sublinear, that is: $p_C(x+y) \le p_C(x) + p_C(y)$ for all $x, y \in C$.

(d) p_C is bounded, that is: there exists $M > 0$ such that $0 \le p_C(x) \le M\|x\|$ for all $x \in C$.

(e) p_C recovers C, that is: $C = \{x \in X : p_C(x) < 1\}$.

Exercise 3 *(Characterization of open unit norm balls in $\mathbb{R}^d$)*

Using the properties of a suitable Minkowski gauge functional, prove the following characterization:

An open set $A \subset \mathbb{R}^d$ is the open unit ball of some norm $\|\cdot\|_*$ on $\mathbb{R}^d$ – that is $A = \{x \in \mathbb{R}^d : \|x\|_* < 1\}$ if and only if $0 \in A$ and A is convex, symmetric[2] and bounded with respect to the standard Euclidean norm.

[1] Recall that $C \subset X$ is convex if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$.
[2] Recall that a linear subspace A is symmetric if $x \in A$ implies that $-x \in A$.

Exercise 4

Consider the following functional on $\mathbb{R}^2$:

$$p(x) = \begin{cases} \sqrt{x_1^2 + x_2^2} & x_1x_2 \geq 0 \\ \max(|x_1|, |x_2|) & x_1x_2 < 0. \end{cases}$$

Using the characterization proved in Exercise 3, discuss whether p is a norm.

Is it possible to define a norm $\|\cdot\|$ on $\mathbb{R}^2$ such that $\|(1,0)\| = \|(0,1)\| = 1$ and $\|(1,1)\| < 1$?

Exercise 5 *(Norms and metrics)*

Let X be a normed linear space and consider the following functional:

$$r(x) := \frac{\|x\|}{1 + \|x\|}, \quad x \in X.$$

(a) Is r a norm on X?

(b) Consider the functional $d \colon X \times X \to \mathbb{R}$ defined by $d(x, y) := r(x - y)$. Prove that d is a metric on X.

(c) Let (x_n) be a sequence in X. Prove that $\|x_n - x\| \to 0$ if and only if $d(x_n, x) \to 0$.

(d) Give an example of a metric in a linear space X that is not associated with a norm.

Exercise 6 *(Distance to a subset)*

Let X be a normed linear space and $A \subseteq X$ a non-empty subset. The distance between $x \in X$ and A is defined as follows:

$$d(x, A) := \inf\{\|x - y\| : y \in A\}.$$

Consider the case where $X = C[0,1]$ and A is the subspace of constant functions on $[0,1]$. Compute the distance $d(f, A)$, where $f(t) = t$.

Exercise 7 *(Compact subsets in ℓ^2)*

Let $(\lambda_n)_n$ be a sequence of positive real numbers. Find conditions on $(\lambda_n)_n$ in such a way that the following subsets of $\ell^2(\mathbb{N})$ are compact:

(a) The parallelepiped $P = \{x = (x_n)_n : |x_n| \leq \lambda_n \, \forall n \in \mathbb{N}\}$.

(b) The ellipsoid $P = \left\{x = (x_n)_n : \sum_n \frac{|x_n|^2}{\lambda_n^2} \leq 1\right\}$.