Analysis (SDS – UNITO, 23/24)

Week 3: more on compactness, Hilbert spaces

S. Ivan Trapasso

Exercise 1 *(Compact vs totally bounded sets)*

Let (X, d) be a metric space. A set $A \subseteq X$ is said to be *totally bounded* if for every $\varepsilon > 0$ there exists a finite number of points $p_1, \ldots, p_N \in A$ such that

$$A \subseteq \bigcup_{j=1}^{N} B(p_j, \varepsilon).$$

Prove the following characterizations.

(a) A subset $A \subseteq X$ is totally bounded if and only if for every $\varepsilon > 0$ there is a compact[1] subset $K \subseteq X$ such that $d(x, K) < \varepsilon$ for every $x \in A$.

(b) A metric space (X, d) is compact if and only if is complete and totally bounded.

Exercise 2 *(Compact subsets in ℓ^2)*

Use the results in Exercise 1 to prove the following result:

A bounded subset $C \subset \ell^p(\mathbb{N})$, $1 \le p < \infty$ is relatively compact[2] if and only if the p-tails of the elements in C vanish uniformly, that is: for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}_+$ such that

$$\sum_{n \ge N_\varepsilon} |x_n|^p < \varepsilon, \quad \forall x = (x_i) \in C.$$

Use this result to solve the following problem:

Let $(\lambda_n)_n$ be a sequence of positive real numbers. Find conditions on $(\lambda_n)_n$ in such a way that the following subsets of $\ell^2(\mathbb{N})$ are compact:

(a) The parallelepiped $P = \{x = (x_n)_n : |x_n| \le \lambda_n \, \forall n \in \mathbb{N}\}$.

(b) The ellipsoid $P = \left\{ x = (x_n)_n : \sum_n \frac{|x_n|^2}{\lambda_n^2} \le 1 \right\}$.

Exercise 3 *(ℓ^p or L^p is Hilbert?)*

Let $1 \le p < \infty$ and $A \subseteq \mathbb{R}^d$ be a Lebesgue measurable set of positive measure.

(a) Prove that $(\ell^p(\mathbb{N}), \| \cdot \|_p)$ is a Hilbert space if and only if $p = 2$.

(b) Prove that $(L^p(A), \| \cdot \|_p)$ is a Hilbert space if and only if $p = 2$.

[1] Recall that $A \subseteq X$ is compact if every open cover of A has a finite subcover.

[2] That is, its closure $\overline{C}$ is compact.

Exercise 4 *(A Hilbert space of matrices)*

Let $M_n(\mathbb{C})$ be the set of all the $n \times n$ complex matrices. Consider the map

$$\langle \cdot, \cdot \rangle \colon M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to \mathbb{C}, \qquad \langle A, B \rangle := \mathrm{Tr}(AB^*).$$

(a) Prove that $(M_n(\mathbb{C}), \langle \cdot, \cdot \rangle)$ is a Hilbert space.

(b) Prove that for all $A, B \in M_n(\mathbb{C})$ we have $|\mathrm{Tr}(AB^*)|^2 \leq \mathrm{Tr}(AA^*)\mathrm{Tr}(BB^*)$.

Exercise 5 *(An inner product space which is not a Hilbert space)*

Let $\ell_0^2(\mathbb{N})$ be the set of complex sequences $x = (x_n)_{n \in \mathbb{N}}$ such that $x_n \neq 0$ only for a finite number of indices $n = 1, \dots, N$.

(a) Prove that $\ell_0^2(\mathbb{N})$ is an inner space product when endowed with the standard ℓ^2 inner product $\langle \cdot, \cdot \rangle$.

(b) Prove that $(\ell_0^2(\mathbb{N}), \langle \cdot, \cdot \rangle)$ is not a Hilbert space.

Exercise 6 *(Orthogonality in real vs complex Hilbert spaces)*

(a) Prove that if $(X, \langle \cdot, \cdot \rangle)$ is a *real* inner product space and $x, y \in X$, if $\|x\|^2 + \|y\|^2 = \|x + y\|^2$ then $\langle x, y \rangle = 0$.

(b) Find an example of a *complex* inner product space $(X, \langle \cdot, \cdot \rangle)$ such that there exist $x, y \in X$ with $\|x\|^2 + \|y\|^2 = \|x + y\|^2$ but $\langle x, y \rangle \neq 0$.

Exercise 7 *(Orthogonality in real vs complex Hilbert spaces)*

(a) Prove that if $(X, \langle \cdot, \cdot \rangle)$ is a *real* inner product space and $x, y \in X$, if $\|x\|^2 + \|y\|^2 = \|x + y\|^2$ then $\langle x, y \rangle = 0$.

(b) Find an example of a *complex* inner product space $(X, \langle \cdot, \cdot \rangle)$ such that there exist $x, y \in X$ with $\|x\|^2 + \|y\|^2 = \|x + y\|^2$ but $\langle x, y \rangle \neq 0$.