Analysis (SDS – UNITO, 23/24)
Weekly mix #5

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Exercise 1

Consider the set

$$C = \{f \in L^2[-1,1] : f(t) = f(-t) \text{ a.e. in } [-1,1]\}.$$

(a) Prove that C is a closed linear subspace of $L^2[-1,1]$.

(b) Determine the explicit form of the orthogonal projection $P_C \colon L^2[-1,1] \to C$.

(c) Determine $C^\perp$.

(d) Compute the distance between $f(t) = t^2 + 1 + \sin(2\pi t)$ and C.

Exercise 2

Let H be a Hilbert space and $\emptyset \neq V \subsetneq H$ be a closed linear subspace of H. Let P_V be the corresponding orthogonal projection.

(a) Find the eigenvalues of P_V and determine the spectrum.

(b) Discuss sufficient conditions on V implying that P_V is a compact operator.

Exercise 3

Let H be a Hilbert space and $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H. Let $T \in \mathcal{B}(H)$ be such that

$$\sum_{n\in\mathbb{N}} \|Te_n\|_H^2 < +\infty.$$

Prove that T is a compact operator.

Exercise 4 *(Spectral properties of the shifts)*

Let $x = (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and consider the operators $S_r, S_l \in \mathcal{B}(\ell^2(\mathbb{N}))$ defined by

$$S_r x = S_r(x_1, x_2, \ldots, x_n, \ldots) \coloneqq (0, x_1, x_2, \ldots x_{n-1}, \ldots),$$

$$S_l x = S_l(x_1, x_2, \ldots, x_n, \ldots) \coloneqq (x_2, x_3, \ldots x_{n+1}, \ldots).$$

(a) Compute the norms $\|S_r\|_{\mathcal{B}}$ and $\|S_l\|_{\mathcal{B}}$.

(b) Discuss whether S_r, S_l are compact operators.

(c) Find $EV(S_r)$ and $\sigma(S_r)$. Determine the eigenspaces, if any.

(d) Find $EV(S_l)$ and $\sigma(S_l)$. Determine the eigenspaces, if any.

(e) Find the adjoint operators S_r^* and S_l^*.

Exercise 5

Consider the operator

$$T\colon L^2(0,1) \to L^2(0,1), \qquad Tf(x) := \int_0^x f(t)dt.$$

(a) Prove that $T \in \mathcal{B}(L^2(0,1))$.

(b) Prove that T is a compact operator.

(c) Find $EV(T)$ and $\sigma(T)$.

(d) Find the adjoint operator T^*.