Analysis (SDS – UNITO, 23/24)
Weekly mix #6

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Exercise 1

Let $a \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$ and $m \in \mathbb{N}$. Prove that the sequence

$$\phi_j(x) \coloneqq e^{-j} j^m a(jx), \qquad j \in \mathbb{N},\, x \in \mathbb{R}^d,$$

converges in $\mathcal{D}(\mathbb{R}^d)$.

Exercise 2

Let $a \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$ and for $j \in \mathbb{N}_+$ consider the sequences

$$\phi_j(x) \coloneqq j^{-1} a(j^{-1}x), \qquad \psi_j(x) \coloneqq j^{-1} a(jx), \qquad x \in \mathbb{R}^d.$$

Discuss the converge of such sequences in $\mathcal{D}(\mathbb{R}^d)$.

Exercise 3

Give sufficient conditions on $N \in \mathbb{R}$ in such a way that $|x|^N \log|x| \in \mathcal{D}'(\mathbb{R}^d)$.

Exercise 4

Prove that $(\log|x|)' = \mathrm{P.V.}\dfrac{1}{x}$ in $\mathcal{D}'(\mathbb{R})$, where the principal value is defined by

$$\Big(\mathrm{P.V.}\frac{1}{x}\Big)(\varphi) \coloneqq \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} dx, \qquad \varphi \in \mathcal{D}(\mathbb{R}).$$

Exercise 5

For $\varepsilon > 0$ consider

$$g_\varepsilon(x) \coloneqq \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}, \qquad x \in \mathbb{R}.$$

Prove that $g_\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \to 0^+$.

Exercise 6

For $\varepsilon > 0$ consider

$$f_\varepsilon(x) \coloneqq (2\pi\varepsilon)^{-d/2} e^{-\frac{|x|^2}{2\varepsilon}}, \qquad x \in \mathbb{R}^d.$$

Prove that $f_\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R}^d)$ as $\varepsilon \to 0^+$.