Analysis (SDS – UNITO, 23/24)
Final simulation

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Notation. $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Question 1

(a) Recall the definition of inner product and inner product space.

(b) Let X be an inner product space and let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in X. Prove Bessel's inequality:

$$\sum_{n\in\mathbb{N}} |(x, e_n)|^2 \leq \|x\|^2, \qquad \forall x \in X.$$

Question 2

Let X, Y be normed spaces and $T \in \mathcal{B}(X, Y)$. Recall the definition of a compact operator and then prove the following statements.

(a) If T has finite rank, then $T \in \mathcal{K}(X, Y)$.

(b) If at least one of the spaces X or Y is finite-dimensional, then $T \in \mathcal{K}(X, Y)$.

Question 3

Let H be a complex Hilbert space. Recall the definition of the spectrum $\sigma(T)$ and the resolvent $\rho(T)$ of an operator $T \in \mathcal{B}(H)$, then prove the following statements.

(a) If $\lambda > \|T\|$ then $\lambda \notin \sigma(T)$.

(b) $\sigma(T)$ is a closed set.

Exercise 4

Consider the linear operator T defined on $\ell^2(\mathbb{N})$ by

$$Tx = \left(\frac{x_1}{1}, \frac{x_2}{8}, \frac{x_3}{27}, \ldots, \frac{x_n}{n^3}, \ldots\right), \qquad x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2(\mathbb{N}).$$

(a) Show that $T \in \mathcal{B}(\ell^2)$ is bounded and compute $\|T\|_{\mathcal{B}(\ell^2)}$.

(b) Prove that $T \in \mathcal{K}(\ell^2)$ and find the eigenvalues of T.

(c) Give the definition of the continuous spectrum of an operator and show that $0 \in \sigma_c(T)$.

Exercise 5

Let $\alpha \in \mathbb{R}$ and consider the operator

$$T\colon \ell^2(\mathbb{N}) \to \mathbb{R}, \qquad Tx := \sum_{n\in\mathbb{N}} x_n e^{\alpha n}, \qquad x = (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N};\mathbb{R}).$$

(a) Find the values of α such that $T \in \mathcal{B}(\ell^2, \mathbb{R}) = (\ell^2)'$.

(b) Compute the norm $\|T\|_{(\ell^2)'}$.

Exercise 6

(a) Show that the Fourier transform of $f(x) = \dfrac{1}{1+x^2}$ in $\mathcal{S}'(\mathbb{R})$ is

$$\hat{f}(\omega) = \int_{\mathbb{R}} \frac{e^{-i\omega x}}{1+x^2} dx = \pi e^{-|\omega|}, \qquad \omega \in \mathbb{R}.$$

(b) Leveraging the previous result, find the Fourier transform of $g(x) = \dfrac{x}{1+x^2}$ in $\mathcal{S}'(\mathbb{R})$.

(c) Leveraging the result in *(a)*, compute the integral

$$\int_0^{+\infty} \frac{1}{(1+x^2)^2} dx.$$