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Some exercises in Set Theory

Exercise 1. Let ZFC_{finite} the axiomatic system ZFC without the infinity axiom. We are going to build a model of ZFC_{finite}.

For $q \in \mathbb{N}$ denote by $[q]$ the unique set of natural numbers $\{p_0, \cdots, p_n\}$ satisfying $q = \sum_{i \leq n} 2^{p_i}$. (So, the elements of $[q]$ are the positions in the binary form of q that are equal to 1.) Define now for $p, q \in \mathbb{N}$ the following relation: pEq when $p \in [q]$.

Show that $(\mathbb{N}, E)$ is a model of ZFC_{finite} and that the infinity axiom is not satisfied in this structure.

Exercise 2. Let ZFC^-_{finite} be the theory ZFC_{finite} without the foundation axiom. We want to show that the foundation axiom is independent of ZFC^-_{finite}.

Given a bijection φ between $\mathbb{N}$ and the set of finite subsets of $\mathbb{N}$, we call $\in_\varphi$ the following relation on $\mathbb{N} \times \mathbb{N}$: $x \in_\varphi y$ if $x \in \varphi(y)$.

1. Show that $(\mathbb{N}, \in_\varphi)$ is a model of ZFC_{finite}.
2. Suppose that for all $p, q \in \mathbb{N}$ we have $p \in_\varphi q$ implies $p < q$. Show that $(\mathbb{N}, \in_\varphi)$ satisfies foundation.
3. Find a φ such that $(\mathbb{N}, \in_\varphi)$ does not satisfy foundation.

Exercise 3. Let MK_{finite} be the theory MK without the infinity axiom, and MK^-_{finite} be the theory MK_{finite} without the foundation axiom.

Call $[\mathbb{N}]^\mathbb{N}$ the set of infinite subsets of natural numbers, and set $M = \mathbb{N} \cup [\mathbb{N}]^\mathbb{N}$.

Given a bijection φ between $\mathbb{N}$ and the set of finite subsets of $\mathbb{N}$, we call $\in_\varphi$ the following relation on $M \times M$: $x \in_\varphi y$ if $x, y \in \mathbb{N}$ and $x \in \varphi(y)$, or if $(x, y) \in \mathbb{N} \times [\mathbb{N}]^\mathbb{N}$ and $x \in y$.

Show that $(M, \in_\varphi)$ is a model of MK^-_{finite} and that the foundation axiom is independant of MK^-_{finite}.

Exercise 4. We consider the following game, for a (finite or infinite) number of participants. We put a hat on the head of each participant, either green or red. Each participant sees the color of all the hats except their own. There is no way for participants to know the color of their own hat. The participants have absolutely no way of communicating during the game, but they have a moment all together, before the game starts, to define a common strategy. They have to guess the color of their hat in one try.

1. There is a finite number of participants. They are linearly ordered at random when the game starts and give their answer in that order. They win the game if only at most one of them guesses wrong. Can they find a winning strategy?

2. There is an infinite number of participants, so they give their answer all at the same time. They win the game if only finitely many of them guess wrong. Can they find a winning strategy? Always?

Exercise 5. Briefly show that ZF proves that the following sets are all equipotent:

$$\mathbb{R}, {}^\omega\omega, {}^\omega 2, {}^\omega\mathbb{R}, {}^\omega({}^\omega\omega), {}^\omega(\omega^2).$$

Exercise 6. Let X be a topological space. A point $x \in X$ is called *isolated* (in X) if the singleton $\{x\}$ is open in X, if not x is called *limit*. Denote by:

$$X' = \{x \in X \mid x \text{ is limit in } X\}.$$

If X is non empty and has no isolated points, we say that X is *perfect*.

1. Show that for each closed set F of X, F' is closed in X.

2. Show that for all closed subsets F of $\mathbb{R}$, the set $F \setminus F'$ of isolated points of F is countable.

   Hint: Recall that $\mathbb{R}$ admits a countable basis of open sets.

   Define the following topological spaces by transfinite induction:

   $$X_0 = X;\ X_{\alpha+1} = (X_\alpha)';\ X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha \text{ for } \lambda \text{ limit}$$

   where each $X_\alpha \subseteq X$ is equipped with the topology induced by X.

3. Show that for each topological space X there exists an ordinal α such that $X_{\alpha+1} = X_\alpha$. The smallest such ordinal, denoted by $CB(X)$, is called the *Cantor-Bendixson rank* of X.

   Hint: Start by showing that the sequence $(X_\alpha)_{\alpha\in Ord}$ is decreasing for inclusion. Then use the replacement axiom and the fact that Ord is a proper class.

4. Show that $X_{CB(X)}$ is either empty or perfect.

   A topological space X such that $X_{CB(X)} = \emptyset$ is called *scattered*. The *weight* of a topological space X, denoted by $w(X)$, is the smallest possible cardinality of a basis for the topology of X.

5. Show that if $w(X) = \kappa$, there does not exist a strictly increasing sequence of open sets (or equivalently, there does not exists a strictly decreasing sequence of closed sets) of length $\geq \kappa^+$, i.e. a sequence $(U_\xi)_{\xi<\gamma}$ with $\gamma \geq \kappa^+$ such that U_ξ is open for all $\xi < \gamma$ and $\xi < \eta < \gamma$ implies $U_\xi \subsetneq U_\eta$.

   Hint: Consider a basis of size κ, proceed by contradiction to prove that the existence of such a sequence would allow the existence of an injection of κ^+ in κ.

6. Show that if $w(X) = \kappa$, then $CB(X) < \kappa$.

7. Show that for each closed subset F of $\mathbb{R}$, $CB(F) < \omega_1$.

8. Show that if a closed subset F of $\mathbb{R}$ is scattered, then it is countable.

Remark. For all successor ordinals $\alpha < \omega_1$ there exists a countable closed subset of $\mathbb{R}$ with $CB(F) = \alpha$. Try to describe it!