Lezione 6 giovedì 14 novembre 2024

**7.8 Theorem** (Assume 7.4) The following are equivalent:

1. $N$ is rich
2. $N$ is homogeneous and universal.

$\Downarrow$ facile

small $<|N|$

$N$ ricco per ogni $k : M \xrightarrow{m} N$ piccolo per ogni $b \in M$ esiste $c \in N$ tale che $k \cup \{\langle b, c \rangle\} : M \xrightarrow{m} N$

$N$ omogeneo per ogni $k : N \xrightarrow{m} N$ piccolo esiste $h : N \xrightarrow{m} N$ totale, suriettivo (nella stessa componente connessa di $N$)

$N$ universale se per ogni $M$ modello, ✓ , $|M| \leq |N|$ esiste $k : M \xhookrightarrow{m} N$

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Esempio $L = \{<\}$ modelli $Mod(T_{lo})$ morfismi i.p

$M = [0, 1]_{\mathbb{Q}}$ $M$ è universale? SI $M$ omogeneo? NO $M$ ricco? NO

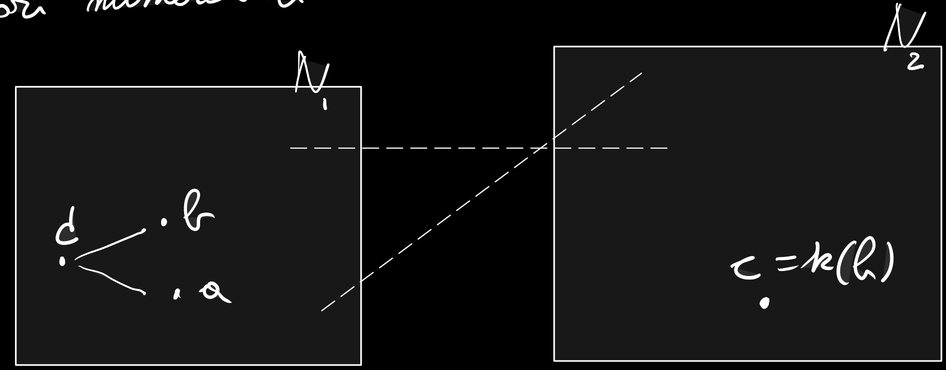
$(0, 1)_{\mathbb{Q}} \models T_{dlo}$

Esempio $L = \{R\}$ $\quad Mod(T_{ga})$, i.p.

$R$ binario

$M = N_1 \sqcup N_2$ $\quad N_i$ grafi aleatori numerabili

$R^M = R^{N_1} \cup R^{N_2}$




M è universale?

M è omogeneo? NO

M è ricco? NO

$\mathrm{dom}\, k = \{a, b\}$

$k(a) = a \qquad k(b) = c$

$k(d) = ?$

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M il grafo con vertici $\omega$ e $R^M = \emptyset$

M è universale? NO

M è omogeneo? SI

M è ricco? NO

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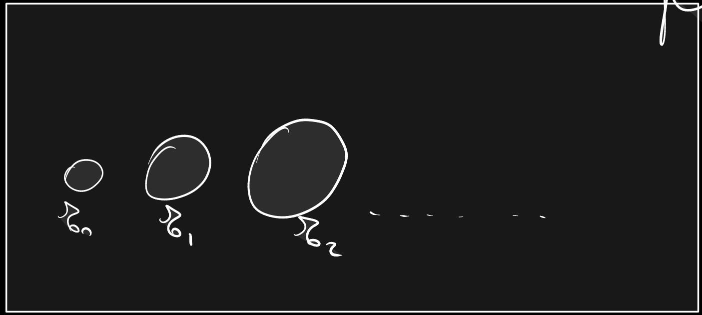
$L = \{R_i : i < \omega\}$

$R_i$ unari

$Mod(T_0)$, i.p.

$$T_0 = \{\exists^{\geq i}_x R_i(x) : i < \omega\} \cup \{\neg \exists x [R_i(x) \wedge R_j(x)] : i < j < \omega\}$$

$M \models T_0$

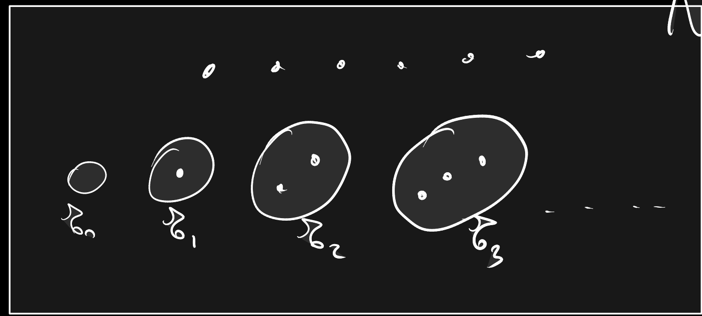


tutti i modelli sono omogenei

Descrivere i modelli universali

$N \models T_0$

è universale e ricco



**7.1 Definition** For ease of reference we list together all properties required below

c1. the (partial) identity map $\mathrm{id}_A : M \to M$ is a morphism, for any $A \subseteq M$

c2. if $k' : M \to N$ is a morphism for all finite $k' \subseteq k$, then $k : M \to N$ is a morphism

c3. morphisms are invertible maps and the inverse of a morphism is a morphism

c4. morphisms preserve the truth of $L_{\mathrm{at}}$-formulas

c5. if $M$ is a model and $N \equiv M$, then also $N$ is a model

c6. every elementary map between models is a morphism.

c7. if $k_i : M \to N$ is a chain of morphisms, then $\bigcup_{i<\lambda} k_i : M \to N$ is a morphism.

**Def**  $M, N$ sono nella stessa componente connessa se $\dfrac{\text{esiste } h: M \xrightarrow{m} N}{\emptyset\ M \xrightarrow{m} N}$

non **Esempio**  $L = \{<, 0, 1\}$  $\mathrm{Mod}(T_{lo})$, i.p.

$$\begin{tikzcd}
 & \mathrm{Mod}(T_{lo}) \arrow[dl] \arrow[dr, bend left] & \\
\mathrm{Mod}(T_{lo} \cup \{0<1\}) & & \mathrm{Mod}(T_{lo} \cup \{1<0\}) \\
M \arrow[u, phantom, "\in" rotate=-90] & & N \arrow[u, phantom, "\in" rotate=-90]
\end{tikzcd}$$

$h: M \xrightarrow{i.p} N$

se  $M \models \varphi(a) \iff N \models \varphi(ha)$  per ogni $\varphi(x) \in L_{at}$

in particolare se $|x| = 0$  $M \models \varphi \iff N \models \varphi$  per ogni $\varphi \in L_{at}$ enunciato

$M \models 0<1 \iff N \models 0<1$

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non Esempio  $L = \{+, -, \cdot, 0, 1\}$  $\mathrm{Mod}(T_{cz})$, i.p.

**Fatto** $M, N$ ricchi $|N| = |M| = \lambda$ $k : M \to N$, $|k| < \lambda$

allora esiste $h : M \xrightarrow{m} N$ totale suriettivo $h \supseteq k$

**Dim** (per andirivieni) induzione TRANSFINITA "ai passi limite prendo l'unione"

$h_{\alpha+1} = h_\alpha$ identico al caso $\omega$

$h = \bigcup_{\alpha < \lambda} h_\alpha$

se $\alpha$ limite $h_\alpha = \bigcup_{\beta < \alpha} h_\beta$

**Corollario** $M$ ricco $\Rightarrow$ $M$ omogeneo $\square$

**Fatto** $N$ ricco, $|M| \leq |N|$, $k : M \xrightarrow{m} N$, $|k| < |N|$

esiste $h : M \overset{m}{\hookrightarrow} N$ $h \supseteq k$.

$M = \{ a_\nu : \nu < \lambda \}$

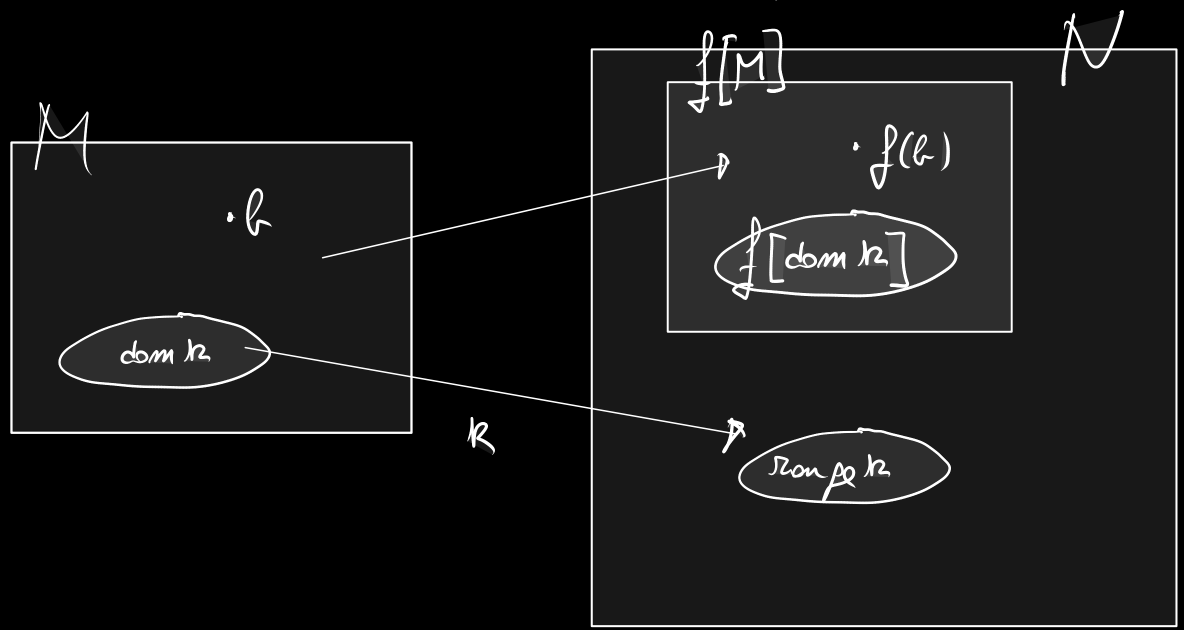
**Dim** mezzo andirivieni

$\lambda \leq |N|$

**Corollario** $M$ ricco $\Rightarrow$ $M$ universale $\square$

<u>Dim</u> $N$ omogeneo + universale $\Longrightarrow$ $N$ ricco

sia $k: M \xrightarrow{m} N$, $|k|<|N|$, $b \in M$ voglio $c \in N$ tale che $k \cup \{\langle b, c\rangle\}: M \xrightarrow{m} N$




esiste $h: M \overset{m}{\hookrightarrow} N$

$k \circ f^{-1}: N \xrightarrow{m} N$

esiste $h: N \xrightarrow{m} N$ totale, suriettivo

$c = h \circ f(b)$ funzione perche $h \circ f \restriction \mathrm{dom}\, k = k$

$k \cup \{\langle b, c\rangle\} = h \circ f \restriction (\mathrm{dom}\, k \cup \{b\})$ è un morfismo. $\square$