AXIOMATIC APPROACH TO HOMOLOGY THEORY

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1. Introduction.—The present paper provides a brief outline of an axiomatic approach to the concept: homology group. It is intended that a full development should appear in book form.

The usual approach to homology theory is by way of the somewhat complicated idea of a complex. In order to arrive at a purely topological concept, the student of the subject is required to wade patiently through a large amount of analytic geometry. Many of the ideas used in the constructions, such as orientation, chain and algebraic boundary, seem artificial. The motivation for their use appears only in retrospect.

Since, in the case of homology groups, the definition by construction is so unwieldy, it is to be expected that an axiomatic approach or definition by properties should result in greater logical simplicity and in a broadened point of view. Naturally enough, the definition by construction is not eliminated by the axiomatic approach. It constitutes an existence proof or proof of consistency.

2. Preliminaries.—The concepts of a topological space and of a group are assumed to be known. The symbol (X, A) stands for a pair consisting of a topological space X and a closed subset A. A map $f:(X, A) \to (Y, B)$ of one such pair into another is a continuous map of X into Y which maps A into B. In case A is the vacuous set (X, A) is written as (X). If f_0, f_1 are two maps of (X, A) into (Y, B), they are homotopic if there exists a homotopy $f(x, t)$ connecting the two maps of X into Y such that $f(x, t) \in B$ for any $x \in A$ and all t.

3. Basic Concepts.—The fundamental concept to be axiomatized is a function $H_q(X, A)$ (called the *q-dimensional, relative homology group of X mod A*) defined for all triples consisting of an integer $q \geqq 0$ and a pair (X, A). The value of the function is an abelian group.

The first subsidiary concept is that of boundary. For each $q \geqq 1$ and each (X, A), there is a homomorphism

$$\partial : H_q(X, A) \to H_{q-1}(A)$$

called the *boundary operator.*

The second subsidiary concept is that of the induced homomorphism. If f is a map of (X, A) into (Y, B) and $q \geqq 0$, there is an attached homomorphism

$$f_*:H_q(X, A) \rightarrow H_q(Y, B)$$

called the *homomorphism induced by f.*

4. *Axioms.*—These three concepts have the following properties.

AXIOM 1. *If f = identity, then f_* = identity.*

That is to say, if f is the identity map of (X, A) on itself, then f_* is the identity map of $H_q(X, A)$ on itself.

AXIOM 2. $(gf)_* = g_*f_*$.

Explicitly, if $f:(X, A) \rightarrow (Y, B)$ and $g:(Y, B) \rightarrow (Z, C)$, then the combination of the induced homomorphisms $f_*:H_q(X, A) \rightarrow H_q(Y, B)$ and $g_*:H_q(Y, B) \rightarrow H_q(Z, C)$ is the induced homomorphism $(gf)_*:H_q(X, A) \rightarrow H_q(Z, C)$.

An immediate consequence of Axioms 1 and 2 is that homeomorphic pairs (X, A) and (Y, B) have isomorphic homology groups.

AXIOM 3. $\partial f_* = f_*\partial$.

Explicitly, if $f:(X, A) \rightarrow (Y, B)$ and $q \geqq 1$, the axiom demands that two homomorphisms of $H_q(X, A)$ into $H_{q-1}(B)$ shall coincide. The first is the combination of $\partial:H_q(X, A) \rightarrow H_{q-1}(A)$ followed by $(f|A)_*: H_{q-1}(A) \rightarrow H_{q-1}(B)$. The second is the combination of $f_*:H_q(X, A) \rightarrow H_q(Y, B)$ followed by $\partial:H_q(Y, B) \rightarrow H_{q-1}(B)$.

AXIOM 4. *If f is homotopic to g, then $f_* = g_*$.*

Definition: The *natural system* of the pair (X, A) is the sequence of groups and homomorphisms

$$\ldots \rightarrow H_q(X) \rightarrow H_q(X, A) \rightarrow H_{q-1}(A) \rightarrow H_{q-1}(X) \rightarrow \ldots \rightarrow H_0(X, A)$$

where $H_q(X) \rightarrow H_q(X, A)$ is induced by the identity map $(X) \rightarrow (X, A)$, $H_q(X, A) \rightarrow H_{q-1}(A)$ is the boundary operation, and $H_{q-1}(A) \rightarrow H_{q-1}(X)$ is induced by the identity map $(A) \rightarrow (X)$.

AXIOM 5. *In the natural system of (X, A) the last group, $H_0(X, A)$, is the image of $H_0(X)$. In any other group of the sequence, the image of the preceding group coincides with the kernel of the succeeding homomorphism.*

At first glance, this axiom may seem strange even to one familiar with homology theory. It is equivalent to three propositions usually stated as follows: (1) the boundary of a cycle of X mod A bounds in A if and only if the cycle is homologous mod A to a cycle of X; (2) a cycle of A is homologous to zero in X if and only if it is the boundary of a cycle of X mod A; (3) a cycle of X is homologous to a cycle of A if and only if it is homologous to zero mod A.

Definition: An open set U of X is strongly contained in A, written $U \Subset A$, if the closure $\overline{U}$ is contained in an open set $V \Subset A$.

AXIOM 6. *If $U \Subset A$, then the identity map: $(X - U, A - U) \rightarrow (X, A)$ induces isomorphisms $H_q(X - U, A - U) \cong H_q(X, A)$ for each $q \geqq 0$.*

This axiom expresses the intuitive idea that $H_q(X, A)$ is pretty much independent of the internal structure of A.

AXIOM 7. *If P is a point, then $H_q(P) = 0$ for $q \geqq 1$.*

A particular reference point P_0 is selected, and $H_0(P_0)$ is called the *coefficient group* of the homology theory.

5. *Uniqueness.*—On the basis of these seven axioms, one can deduce the entire homology theory of a complex in the usual sense. Some highlights of the procedure are the following.

If σ is an n-simplex, and $\dot{\sigma}$ its point-set boundary, then $H_n(\sigma, \dot{\sigma})$ is isomorphic to the coefficient group. Further, $H_q(\sigma, \dot{\sigma}) = 0$ for $q \neq n$, and the boundary operator $\partial: H_n(\sigma, \dot{\sigma}) \to H_{n-1}(\dot{\sigma})$ is an isomorphism onto for $n > 1$, and into for $n = 1$.

Let f be the simplicial map of σ on itself which interchanges two vertices and leaves all others fixed. Then, for any $z \in H_n(\sigma, \dot{\sigma})$, we have $f_*(z) = -z$. This permits the usual division of permutations into the classes of even and odd, and leads naturally to a definition of orientation—a concept which is quite troublesome in the usual approach.

Definition: Let H, H' be two homology theories satisfying Axioms 1 through 7. A *homomorphism*

$$h: H \to H'$$

is defined to be a system of homomorphisms

$$h(q, X, A): H_q(X, A) \to H_q'(X, A)$$

defined for all q, (X, A), which commute properly with the boundary operator and induced homomorphisms:

$$h(q-1, A)\partial = \partial' h(q, X, A), \qquad h(q, Y, B)f_* = f_*' h(q, X, A). \qquad \text{(I)}$$

If h gives an isomorphism of the coefficient groups $h(0, P_0): H_0(P_0) \cong H_0'(P_0)$, then h is called a *strong homomorphism.* If each $h(q, X, A)$ is an isomorphism, then h is called an *equivalence* and H and H' are called *equivalent.*

Since the usual homology theory of complexes is deducible from the axioms, there follows the

UNIQUENESS THEOREM: *Any two homology theories having the same coefficient group coincide on complexes.*

Explicitly, if $i: H_0(P_0) \cong H_0'(P_0)$ is an isomorphism between the coefficient groups of H and H', then isomorphisms

$$h(q, X, A): H_q(X, A) \cong H_q'(X, A)$$

can be defined for X a complex, A a subcomplex such that $h(0, P_0)$ coincides with i, and the relations (I) hold in so far as they are defined (f need not be simplicial). Indeed, there is just one way of constructing $h(q, X, A)$. The uniqueness theorem implies that any strong homomorphism $h: H \to$

H' is an equivalence as far as complexes are concerned. In view of Axiom 4, the uniqueness theorem holds for spaces having the same homotopy type as complexes. These include the absolute neighborhood retracts.

6. *Existence.*—As is to be expected, homology theories exist which satisfy the axioms. Both the Čech homology theory H^1 and the singular homology theory H^0 satisfy the axioms. This is fairly well known, although the proofs of some of the axioms are only implicitly contained in the literature. It is well known that the two homology theories differ for some pairs (X, A). Thus, the axioms do not provide uniqueness for all spaces.

The surprising feature of H^0 and H^1 that appears in this development is that they play extreme roles in the family of all homology theories, and have parallel definitions. They can be defined as follows: The homology groups of the simplicial structure of a complex (using chains, etc.) are defined as usual. (As a first step of an existence proof, this is quite natural since the definition has been deduced from the axioms.) Using maps $K \rightarrow X$ of complexes into the space X, the singular homology groups $H^0{}_q(X, A)$ can be defined using a suitable limiting process. Similarly using maps $X \rightarrow K$ of the space into complexes, the Čech homology groups $H^1{}_q(X, A)$ are obtained. It is then established that H^0 and H^1 are minimal and maximal in the family of all homology theories with a prescribed coefficient group in the sense that, if H is any homology theory, there exist strong homomorphisms $H^0 \rightarrow H \rightarrow H^1$. This is an indication of how it is possible to characterize H^0 or H^1 by the addition of a suitable Axiom 8.

7. *Generalizations.*—A suitable refinement of the axioms will permit the introduction of topologized homology groups.

Cohomology can be axiomatized in the same way as homology. It is only necessary to reverse the directions of the operators ∂ and f_* in the above axioms and make such modifications in the statements as these reversals entail. The analogous uniqueness theorems can be proved.

The products of elements of two cohomology groups with values in a third (in the usual sense) may also be axiomatized and characterized uniquely.

AWARDS TO THE ACADEMY AND RESEARCH COUNCIL

The National Academy of Sciences was incorporated by an act of the Congress in 1863 during the Civil War to mobilize the best scientific talent of the country for service to the Government in war and in peace. The National Research Council was organized in 1916 at the request of the President by the National Academy of Sciences as a measure of national