Proof. After a linear transformation (over K which is algebraically closed), we can effectively assume that $Q(x_0, x_1, x_2, x_3) = x_0x_1 - x_2x_3$ (this amounts to writing the space as a direct sum of two hyperbolic planes). The isomorphism of the quadric to $\mathbf{P}^1 \times \mathbf{P}^1$ is thus a particular case of Proposition B-1.6. □

The following two lemmas are special cases of Bézout's theorem, proven further down (Theorem B-2.4).

1.13. Lemma. *Let C be a curve of degree d (i.e., defined by a homogeneous polynomial of degree d) in the projective plane and not containing the line D of $\mathbf{P}^2$. Then $C \cap D$ is composed of d points (counted with multiplicity).*

Proof. Let $F(x_0, x_1, x_2) = 0$ be the equation of degree d of C and $a_0x_0 + a_1x_1 + a_2x_2 = 0$ that of D. One of the a_i is non-zero, so we can take it to be a_0. The equation of points of intersection of C and D is therefore written $x_0 = -\frac{a_1}{a_0}x_1 - \frac{a_2}{a_0}x_2$ and

$$F\left(-\frac{a_1}{a_0}x_1 - \frac{a_2}{a_0}x_2, x_1, x_2\right) = 0,$$

which factors as $a\prod_i(\alpha_i x_1 - \beta_i x_2)^{m_i}$ with $\sum_i m_i = d$. □

1.14. Lemma. *If C is a curve of degree d in the projective plane with no components in common with the conic D of $\mathbf{P}^2$, then $C \cap D$ is composed of $2d$ points (counted with multiplicity).*

Proof. If the conic is composed of two lines, this lemma can be deduced from the previous lemma. We can thus assume that the conic is irreducible. Up to a linear change of coordinates, we can assume that the conic is written as $x_1x_0 - x_2^2 = 0$ and hence that it is parametrized by the map from $\mathbf{P}^1$ to $\mathbf{P}^2$ given by $(y_0, y_1) \mapsto (y_0^2, y_1^2, y_0y_1)$. Let $F(x_0, x_1, x_2) = 0$ be the equation of C. The equation of the points of intersection of C and D is thus written $P = (y_0^2, y_1^2, y_0y_1)$ and

$$F\left(y_0^2, y_1^2, y_0y_1\right) = 0,$$

which factors into $a\prod_i(\alpha_i y_1 - \beta_i y_0)^{m_i}$ with $\sum_i m_i = 2d$. □

Notation. We denote by $S_{n,d}$ the vector space of homogeneous polynomials of degree d in $x_0, \ldots, x_n$, and if $P_1, \ldots, P_r$ are points of $\mathbf{P}^n$, we denote by $S_{n,d}(P_1, \ldots, P_r)$ the subspace of $S_{n,d}$ formed of polynomials which vanish at each P_i.

1.15. Definition. A *linear system of hypersurfaces S* of degree d in $\mathbf{P}^n$ is a vector subspace S of $S_{n,d}$.

The set of hypersurfaces corresponding to the polynomials of S can be seen as a linear subvariety of dimension $\dim(S) - 1$ in the projective space corresponding to $S_{n,d}$.

1.16. Lemma. *We have the following formulas:*

$$\dim S_{n,d} = \binom{n+d}{d} \qquad \textit{and} \qquad \dim S_{n,d}(P_1, \ldots, P_r) \geqslant \dim S_{n,d} - r.$$

The lemma is obvious by noticing that vanishing at point P is a linear condition on the coefficients of a polynomial. The computation of the exact dimension of $S_{n,d}(P_1, \ldots, P_r)$ can however be tricky.

1.17. Examples. We have

$\dim S_{2,d} = \dfrac{(d+2)(d+1)}{2}$ and $\dim S_{2,d}(P_1, \ldots, P_r) \geqslant \dfrac{(d+2)(d+1)}{2} - r$

and, in particular, $\dim S_{2,2} = 6$ and $\dim S_{2,2}(P_1, \ldots, P_r) \geqslant 6 - r$. Thus there always passes at least one conic through any five given points. We can specify under which conditions such a conic is unique.

1.18. Lemma. *Through any five points $P_1, \ldots, P_5$ in the projective plane, there always passes a conic. Furthermore, if no four of the points are colinear, the conic is unique, i.e.,* $\dim S_{2,2}(P_1, \ldots, P_5) = 1$.

Proof. We will first treat the case where three of the points, P_1, P_2, P_3, are colinear. The conic must contain the line $L = 0$ defined by the three points. Hence, we have $S_{2,2}(P_1, \ldots, P_5) = LS_{2,1}(P_4, P_5)$ since P_4 and P_5 are not on the line $L = 0$. There is only one line which passes through P_4 and P_5, hence $\dim S_{2,1}(P_4, P_5) = 1$ and $\dim S_{2,2}(P_1, \ldots, P_5) = 1$. We will now treat the case where no three of the P_i are colinear. Suppose $\dim S_{2,2}(P_1, \ldots, P_5) > 1$, and let P_6 be a point distinct from P_4 and P_5 on the line $L = 0$ defined by these two points. We would then have $\dim S_{2,2}(P_1, \ldots, P_6) \geqslant 1$, and a corresponding conic containing P_4, P_5, P_6 must contain the whole line hence be composed of two lines, and then P_1, P_2, P_3 would be colinear. □

The dimension of $S_{2,3}$ is 10. Therefore, there is always a cubic passing through any nine points in the projective plane plane. If 4 of these points are colinear, the cubic must contain the corresponding line, and if 7 of these points are on the same conic, the cubic must contain the corresponding conic.

1.19. Definition. A point $P = (x_0, \ldots, x_n)$ on a hypersurface $V = \{P \in$