Now we prove the uniform continuity theorem of calculus. In the process, we are led to introduce a new notion that will prove to be surprisingly useful, that of a *Lebesgue number* for an open covering of a metric space. First, a preliminary notion:

Definition. Let (X, d) be a metric space; let A be a nonempty subset of X. For each $x \in X$, we define the ***distance from x to A*** by the equation

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

It is easy to show that for fixed A, the function $d(x, A)$ is a continuous function of x: Given $x, y \in X$, one has the inequalities

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a),$$

for each $a \in A$. It follows that

$$d(x, A) - d(x, y) \leq \inf d(y, a) = d(y, A),$$

so that

$$d(x, A) - d(y, A) \leq d(x, y).$$

The same inequality holds with x and y interchanged; continuity of the function $d(x, A)$ follows.

Now we introduce the notion of Lebesgue number. Recall that the *diameter* of a bounded subset A of a metric space (X, d) is the number

$$\sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}.$$

Lemma 27.5 (The Lebesgue number lemma). *Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it.*

The number δ is called a ***Lebesgue number*** for the covering $\mathcal{A}$.

Proof. Let $\mathcal{A}$ be an open covering of X. If X itself is an element of $\mathcal{A}$, then any positive number is a Lebesgue number for $\mathcal{A}$. So assume X is not an element of $\mathcal{A}$.

Choose a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers X. For each i, set $C_i = X - A_i$, and define $f : X \to \mathbb{R}$ by letting $f(x)$ be the average of the numbers $d(x, C_i)$. That is,

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

We show that $f(x) > 0$ for all x. Given $x \in X$, choose i so that $x \in A_i$. Then choose ϵ so the ϵ-neighborhood of x lies in A_i. Then $d(x, C_i) \geq \epsilon$, so that $f(x) \geq \epsilon/n$.

Since f is continuous, it has a minimum value δ; we show that δ is our required Lebesgue number. Let B be a subset of X of diameter less than δ. Choose a point x_0 of B; then B lies in the δ-neighborhood of x_0. Now

$$\delta \leq f(x_0) \leq d(x_0, C_m),$$

where $d(x_0, C_m)$ is the largest of the numbers $d(x_0, C_i)$. Then the δ-neighborhood of x_0 is contained in the element $A_m = X - C_m$ of the covering $\mathcal{A}$. ∎

Definition. A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be ***uniformly continuous*** if given $\epsilon > 0$, there is a $\delta > 0$ such that for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \Longrightarrow d_Y(f(x_0), f(x_1)) < \epsilon.$$

Theorem 27.6 (Uniform continuity theorem). *Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.*

Proof. Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \epsilon/2)$ of radius $\epsilon/2$. Let $\mathcal{A}$ be the open covering of X by the inverse images of these balls under f. Choose δ to be a Lebesgue number for the covering $\mathcal{A}$. Then if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$, the two-point set $\{x_1, x_2\}$ has diameter less than δ, so that its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \epsilon/2)$. Then $d_Y(f(x_1), f(x_2)) < \epsilon$, as desired. ∎

Finally, we prove that the real numbers are uncountable. The interesting thing about this proof is that it involves no algebra at all—no decimal or binary expansions of real numbers or the like—just the order properties of $\mathbb{R}$.

Definition. If X is a space, a point x of X is said to be an ***isolated point*** of X if the one-point set $\{x\}$ is open in X.

Theorem 27.7. *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.*

Proof. Step 1. We show first that given any nonempty open set U of X and any point x of X, there exists a nonempty open set V contained in U such that $x \notin \bar{V}$.

Choose a point y of U different from x; this is possible if x is in U because x is not an isolated point of X and it is possible if x is not in U simply because U is nonempty. Now choose disjoint open sets W_1 and W_2 about x and y, respectively. Then the set $V = W_2 \cap U$ is the desired open set; it is contained in U, it is nonempty because it contains y, and its closure does not contain x. See Figure 27.3.

Step 2. We show that given $f : \mathbb{Z}_+ \to X$, the function f is not surjective. It follows that X is uncountable.