of doing this. One can use only techniques of algebra; this proof is long and arduous. Or one can develop the theory of analytic functions of a complex variable to the point where it becomes a trivial corollary of Liouville's theorem. Or one can prove it as a relatively easy corollary of our computation of the fundamental group of the circle; this we do now.

Theorem 56.1 (The fundamental theorem of algebra). *A polynomial equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree $n > 0$ with real or complex coefficients has at least one (real or complex) root.

Proof. *Step 1.* Consider the map $f : S^1 \to S^1$ given by $f(z) = z^n$, where z is a complex number. We show that the induced homomorphism f_* of fundamental groups is injective.

Let $p_0 : I \to S^1$ be the standard loop in S^1,

$$p_0(s) = e^{2\pi i s} = (\cos 2\pi s, \sin 2\pi s).$$

Its image under f_* is the loop

$$f(p_0(s)) = (e^{2\pi i s})^n = (\cos 2\pi n s, \sin 2\pi n s).$$

This loop lifts to the path $s \to ns$ in the covering space $\mathbb{R}$. Therefore, the loop $f \circ p_0$ corresponds to the integer n under the standard isomorphism of $\pi_1(S^1, b_0)$ with the integers, whereas p_0 corresponds to the number 1. Thus f_* is "multiplication by n" in the fundamental group of S^1, so that in particular, f_* is injective.

Step 2. We show that if $g : S^1 \to \mathbb{R}^2 - \mathbf{0}$ is the map $g(z) = z^n$, then g is not nulhomotopic.

The map g equals the map f of Step 1 followed by the inclusion map $j : S^1 \to \mathbb{R}^2 - \mathbf{0}$. Now f_* is injective, and j_* is injective because S^1 is a retract of $\mathbb{R}^2 - \mathbf{0}$. Therefore, $g_* = j_* \circ f_*$ is injective. Thus g cannot be nulhomotopic.

Step 3. Now we prove a special case of the theorem. Given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

we assume that

$$|a_{n-1}| + \cdots + |a_1| + |a_0| < 1$$

and show that the equation has a root lying in the unit ball B^2.

Assume it has no such root. Then we can define a map $k : B^2 \to \mathbb{R}^2 - \mathbf{0}$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

Let h be the restriction of k to S^1. Because h extends to a map of the unit ball into $\mathbb{R}^2 - \mathbf{0}$, the map h is nulhomotopic.

On the other hand, we shall define a homotopy F between h and the map g of Step 2; since g is not nulhomotopic, we have a contradiction. We define $F : S^1 \times I \to \mathbb{R}^2 - \mathbf{0}$ by the equation

$$F(z, t) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0).$$

See Figure 56.1; $F(z, t)$ never equals $\mathbf{0}$ because

$$\begin{aligned} |F(z,t)| &\geq |z^n| - |t(a_{n-1}z^{n-1} + \cdots + a_0)| \\ &\geq 1 - t(|a_{n-1}z^{n-1}| + \cdots + |a_0|) \\ &= 1 - t(|a_{n-1}| + \cdots + |a_0|) > 0. \end{aligned}$$

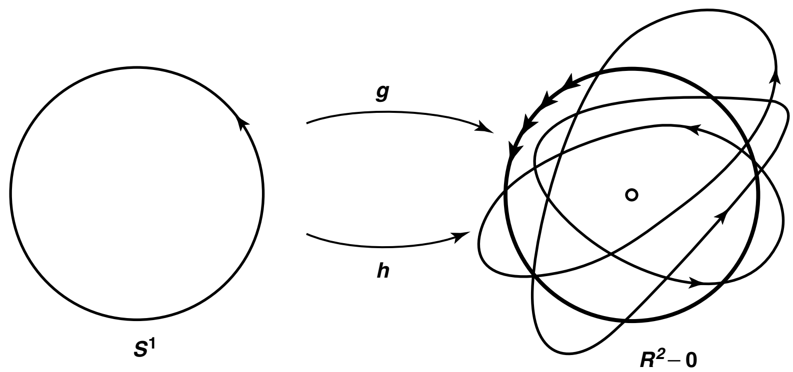

Figure 56.1

Step 4. Now we prove the general case. Given a polynomial equation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

let us choose a real number $c > 0$ and substitute $x = cy$. We obtain the equation

$$(cy)^n + a_{n-1}(cy)^{n-1} + \cdots + a_1(cy) + a_0 = 0$$

or

$$y^n + \frac{a_{n-1}}{c}y^{n-1} + \cdots + \frac{a_1}{c^{n-1}}y + \frac{a_0}{c^n} = 0.$$

If this equation has the root $y = y_0$, then the original equation has the root $x_0 = cy_0$. So we need merely choose c large enough that

$$\left|\frac{a_{n-1}}{c}\right| + \left|\frac{a_{n-2}}{c^2}\right| + \cdots + \left|\frac{a_1}{c^{n-1}}\right| + \left|\frac{a_0}{c^n}\right| < 1$$

to reduce the theorem to the special case considered in Step 3. ■