Since $ds/dt = |\alpha'(t)| \neq 0$, the function $s = s(t)$ has a differentiable inverse $t = t(s)$, $s \in s(I) = J$, where, by an abuse of notation, t also denotes the inverse function s^{-1} of s. Now set $\beta = \alpha \circ t\colon J \to R^3$. Clearly, $\beta(J) = \alpha(I)$ and $|\beta'(s)| = |(\alpha'(t) \cdot (dt/ds)| = 1$. This shows that β has the same trace as α and is parametrized by arc length. It is usual to say that β is a *reparametrization of* $\alpha(I)$ *by arc length.*

This fact allows us to extend all local concepts previously defined to regular curves with an arbitrary parameter. Thus, we say that the curvature $k(t)$ of $\alpha\colon I \to R^3$ at $t \in I$ is the curvature of a reparametrization $\beta\colon J \to R^3$ of $\alpha(I)$ by arc length at the corresponding point $s = s(t)$. This is clearly independent of the choice of β and shows that the restriction, made at the end of Sec. 1-3, of considering only curves parametrized by arc length is not essential.

In applications, it is often convenient to have explicit formulas for the geometrical entities in terms of an arbitrary parameter; we shall present some of them in Exercise 12.

EXERCISES

Unless explicity stated, α: I $\to$ R^3 *is a curve parametrized by arc length* s, *with curvature* k(s) $\neq 0$, *for all* s $\in$ I.

1. Given the parametrized curve (helix)

$$\alpha(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \quad s \in R,$$

where $c^2 = a^2 + b^2$,

- **a.** Show that the parameter s is the arc length.
- **b.** Determine the curvature and the torsion of α.
- **c.** Determine the osculating plane of α.
- **d.** Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.
- **e.** Show that the tangent lines to α make a constant angle with the z axis.

***2.** Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}.$$

3. Assume that $\alpha(I) \subset R^2$ (i.e., α is a plane curve) and give k a sign as in the text. Transport the vectors $t(s)$ parallel to themselves in such a way that the origins of $t(s)$ agree with the origin of R^2; the end points of $t(s)$ then describe a parametrized curve $s \to t(s)$ called the *indicatrix* *of tangents* of α. Let $\theta(s)$ be the angle from e_1 to $t(s)$ in the orientation of R^2. Prove (a) and (b) (notice that we are assuming that $k \neq 0$).

a. The indicatrix of tangents is a regular parametrized curve.

b. $dt/ds = (d\theta/ds)n$, that is, $k = d\theta/ds$.

***4.** Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.

5. A regular parametrized curve α has the property that all its tangent lines pass through a fixed point.

a. Prove that the trace of α is a (segment of a) straight line.

b. Does the conclusion in part a still hold if α is not regular?

6. A *translation* by a vector v in R^3 is the map A: $R^3 \to R^3$ that is given by $A(p) = p + v$, $p \in R^3$. A linear map ρ: $R^3 \to R^3$ is an *orthogonal transformation* when $\rho u \cdot \rho v = u \cdot v$ for all vectors u, $v \in R^3$. A *rigid motion* in R^3 is the result of composing a translation with an orthogonal transformation with positive determinant (this last condition is included because we expect rigid motions to preserve orientation).

a. Demonstrate that the norm of a vector and the angle θ between two vectors, $0 \leq \theta \leq \pi$, are invariant under orthogonal transformations with positive determinant.

b. Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?

c. Show that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

***7.** Let α: $I \to R^2$ be a regular parametrized plane curve (arbitrary parameter), and define $n = n(t)$ and $k = k(t)$ as in Remark 1. Assume that $k(t) \neq 0$, $t \in I$. In this situation, the curve

$$\beta(t) = \alpha(t) + \frac{1}{k(t)} n(t), \quad t \in I,$$

is called the *evolute* of α (Fig. 1-17).

a. Show that the tangent at t of the evolute of α is the normal to α at t.

b. Consider the normal lines of α at two neighboring points t_1, t_2, $t_1 \neq t_2$. Let t_1 approach t_2 and show that the intersection points of the normals converge to a point on the trace of the evolute of α.

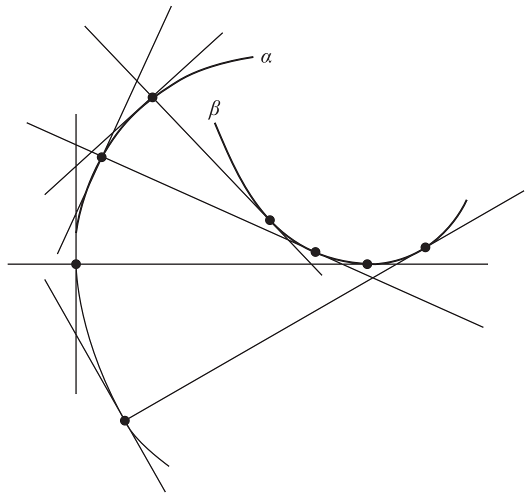

Figure 1-17

8. The trace of the parametrized curve (arbitrary parameter)

$$\alpha(t) = (t, \cosh t), \quad t \in R,$$

is called the *catenary*.

a. Show that the signed curvature (cf. Remark 1) of the catenary is

$$k(t) = \frac{1}{\cosh^2 t}.$$

b. Show that the evolute (cf. Exercise 7) of the catenary is

$$\beta(t) = (t - \sinh t \cosh t, 2\cosh t).$$

9. Given a differentiable function $k(s)$, $s \in I$, show that the parametrized plane curve having $k(s) = k$ as curvature is given by

$$\alpha(s) = \left(\int \cos\theta(s)\, ds + a, \int \sin\theta(s)\, ds + b\right),$$

where

$$\theta(s) = \int k(s)\, ds + \varphi,$$

and that the curve is determined up to a translation of the vector (a, b) and a rotation of the angle φ.

10. Consider the map

$$\alpha(t) = \begin{cases} (t, 0, e^{-1/t^2}), & t > 0 \\ (t, e^{-1/t^2}, 0), & t < 0 \\ (0, 0, 0), & t = 0 \end{cases}$$

a. Prove that α is a differentiable curve.

b. Prove that α is regular for all t and that the curvature $k(t) \neq 0$, for $t \neq 0$, $t \neq \pm\sqrt{2/3}$, and $k(0) = 0$.

c. Show that the limit of the osculating planes as $t \to 0$, $t > 0$, is the plane $y = 0$ but that the limit of the osculating planes as $t \to 0$, $t < 0$, is the plane $z = 0$ (this implies that the normal vector is discontinuous at $t = 0$ and shows why we excluded points where $k = 0$).

d. Show that τ can be defined so that $\tau \equiv 0$, even though α is not a plane curve.

11. One often gives a plane curve in polar coordinates by $\rho = \rho(\theta)$, $a \leq \theta \leq b$.

a. Show that the arc length is

$$\int_a^b \sqrt{\rho^2 + (\rho')^2}\, d\theta,$$

where the prime denotes the derivative relative to θ.

b. Show that the curvature is

$$k(\theta) = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\{(\rho')^2 + \rho^2\}^{3/2}}.$$

12. Let $\alpha: I \to R^3$ be a regular parametrized curve (not necessarily by arc length) and let $\beta: J \to R^3$ be a reparametrization of $\alpha(I)$ by the arc length $s = s(t)$, measured from $t_0 \in I$ (see Remark 2). Let $t = t(s)$ be the inverse function of s and set $d\alpha/dt = \alpha'$, $d^2\alpha/dt^2 = \alpha''$, etc. Prove that

a. $dt/ds = 1/|\alpha'|$, $d^2t/ds^2 = -(\alpha' \cdot \alpha''/|\alpha'|^4)$.

b. The curvature of α at $t \in I$ is

$$k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}.$$

c. The torsion of α at $t \in I$ is

$$\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

d. If $\alpha\colon I \to R^2$ is a plane curve $\alpha(t) = (x(t), y(t))$, the signed curvature (see Remark 1) of α at t is

$$k(t) = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}}.$$

***13.** Assume that $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all $s \in I$. Show that a necessary and sufficient condition for $\alpha(I)$ to lie on a sphere is that

$$R^2 + (R')^2 T^2 = \text{const.},$$

where $R = 1/k$, $T = 1/\tau$, and R' is the derivative of R relative to s.

14. Let $\alpha\colon (a, b) \to R^2$ be a regular parametrized plane curve. Assume that there exists t_0, $a < t_0 < b$, such that the distance $|\alpha(t)|$ from the origin to the trace of α will be a maximum at t_0. Prove that the curvature k of α at t_0 satisfies $|k(t_0)| \geq 1/|\alpha(t_0)|$.

***15.** Show that the knowledge of the vector function $b = b(s)$ (binormal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $k(s)$ and the absolute value of the torsion $\tau(s)$ of α.

***16.** Show that the knowledge of the vector function $n = n(s)$ (normal vector) of a curve α, with nonzero torsion everywhere, determines the curvature $k(s)$ and the torsion $\tau(s)$ of α.

17. In general, a curve α is called a *helix* if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$, and prove that:

***a.** α is a helix if and only if $k/\tau = \text{const.}$

***b.** α is a helix if and only if the lines containing $n(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

***c.** α is a helix if and only if the lines containing $b(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

d. The curve

$$\alpha(s) = \left(\frac{a}{c} \int \sin\theta(s)\, ds,\ \frac{a}{c} \int \cos\theta(s)\, ds,\ \frac{b}{c} s \right),$$

where $c^2 = a^2 + b^2$, is a helix, and that $k/\tau = a/b$.

***18.** Let $\alpha\colon I \to R^3$ be a parametrized regular curve (not necessarily by arc length) with $k(t) \neq 0$, $\tau(t) \neq 0$, $t \in I$. The curve α is called a *Bertrand curve* if there exists a curve $\bar{\alpha}\colon I \to R^3$ such that the normal lines of α and $\bar{\alpha}$ at $t \in I$ are equal. In this case, $\bar{\alpha}$ is called a *Bertrand mate* of α, and we can write

$$\bar{\alpha}(t) = \alpha(t) + r n(t).$$