A BRIEF REPORT ON JOHN MILNOR'S BRIEF EXCURSIONS INTO DIFFERENTIAL GEOMETRY

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Introduction

This lecture supplements the last two afternoons' presentations, jointly covering Milnor's vast contributions to differential and algebraic topology. Though it deals with only a handful of Milnor's papers, those that belong to the field of differential geometry, rather than topology, this selection proves to be quite rewarding. Among other things, it includes an item of special interest in this retrospective: Milnor's very first paper.

This famous paper, written when Milnor was a freshman at Princeton, has already been alluded to by many other speakers. But now it will be examined a little more thoroughly; in fact, in contrast to previous lectures, in this talk we will actually give a proof of a theorem. So perhaps I had better first allay the fears of those blissfully ignorant of the subject, by mentioning one beautiful aspect of this famous paper of differential geometry: it doesn't really require knowing any differential geometry at all.

*That the author of this paper should also be the publisher of these Proceedings is but one strange aspect of this lecture. I have tried to retain the informal nature of the talk, which was meant both as an antidote to the unremitting high-powered mathematics presented throughout the symposium, and as a tribute to the delightfully informal nature of all of Milnor's own lectures.

1. Milnor's first theorem

Let's begin by considering a smooth curve $C\colon [0, L] \to \mathbf{R}^3$ that is *closed*, meaning that $C(0) = C(L)$ and $C'(0) = C'(L)$.

We might as well assume that C is "parameterized by arc-length", so that $|C'(s)| = 1$ for all s; we use $\mathbf{t}(s) = C'(s)$ for this unit tangent vector. Then the derivative $C''(s) = \mathbf{t}'(s)$ represents the rate at which the curve deviates from being a straight line, and the norm of this derivative, $\kappa(s) = |C''(s)|$ is called the *curvature* of C at s. Finally, the integral of this curvature,

$$\int_0^L \kappa(s)\,ds = \int_0^L |C''(s)|\,ds$$

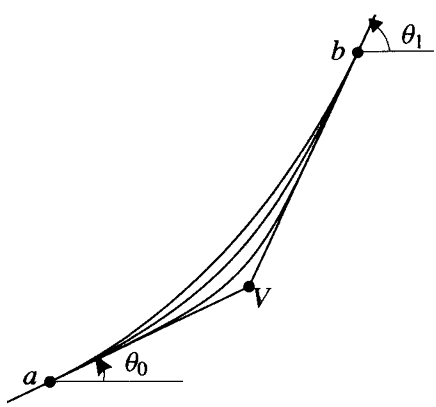
is called the *total curvature* $\kappa(C)$ of the curve C.

Milnor's theorem states:

THEOREM. If C is knotted, then the total curvature $\kappa(C)$ satisfies

$$\kappa(C) \geq 4\pi.$$

The definition of κ was made for smooth curves, but we can easily extend it to a piecewise smooth curve, and in particular to a polygonal curve, by considering a sequence of smooth curves that approaches the polygonal curve P. The figure below shows such a sequence near a vertex V of the polygonal curve.



In this situation we can write

$$C'(t) = (\cos\theta(t), \sin\theta(t)),$$

where θ is nondecreasing, so

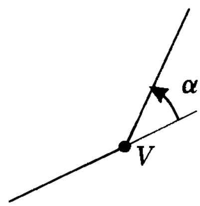
$$C''(t) = \theta'(t) \cdot (-\sin\theta(t), \cos\theta(t))$$
$$|C''(t)| = \theta'(t)$$

and consequently

$$\int_a^b |C''(s)|\, ds = \theta_1 - \theta_0.$$

It follows that as $a, b \to V$, the integral

$$\int_a^b |C''(s)|\, ds \to \alpha$$



where α is the exterior angle at the vertex v. Consequently,

$$\kappa(P) = \sum \text{exterior angles of } P.$$

Conversely, if we use this as a definition of $\kappa(P)$ for polygonal curves P, then by considering all polygonal curves $P \prec C$ inscribed in a smooth curve C, we can define $\kappa(C)$ as

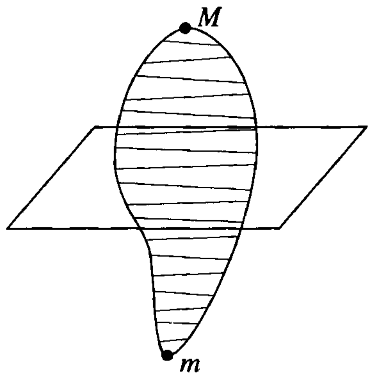
$$\kappa(C) = \sup_{P \prec C} \kappa(P).$$

(To be sure, one has to check that this definition really coincides with the original definition, by playing around with the mean-value theorem appropriately.)

One other preparatory remark is required. For any plane γ, we can consider the function that gives the height of C above γ, which is just

$$s \mapsto C(s) \cdot X \qquad \text{for a unit vector } X \perp \gamma.$$

Suppose that for some plane γ this function happens to have a *unique* local maximum, at M, and let m be a minimum point for this function, as in the figure below.



Then the curve C is divided into two arcs joining m and M, and it is easy to see that any plane parallel to γ between m and M must intersect each of these arcs just once, since M is a unique local maximum for the height function. If we now join these unique points on the two arcs with straight lines, we obtain a disk bounded by C. It follows, in particular, that C is not knotted.

The proof of Milnor's theorem will proceed by showing that

$$\kappa(C) < 4\pi \implies \exists X : \text{the function } C(s)\cdot X \text{ has a unique local maximum.}$$

Now how do we go about proving such a result? Or, more precisely, how did Milnor go about proving it? As we might expect, by an ingeniously indirect method.

For any unit vector X, let $\mu_C(X)$ be the number of local maxima of $C(s)\cdot X$,

$$\mu_C(X) = \#\{\text{local maxima of } C(s)\cdot X\}.$$

Then we will prove the following:

$$(*) \qquad \int_{S^2} \mu_C \, dA = 2\kappa(C).$$

Once we have this formula we will be done, for then

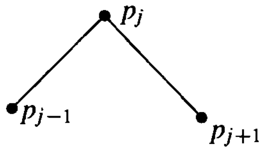
$$\kappa(C) < 4\pi \implies \int_{S^2} \mu_C \, dA < 8\pi = 2\int_{S^2} dA$$

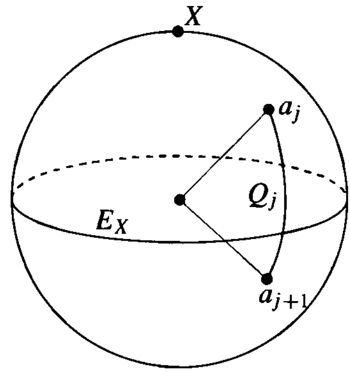
$$\implies \mu_C(X) < 2 \quad \text{for some } X \in S^2.$$

To prove $(*)$, we first consider a polygonal curve P with vertices p_0, p_1, $\dots$, p_{n-1}, $p_n = p_0$. These vertices yield n points

$$a_j = \frac{p_j - p_{j-1}}{|p_j - p_{j-1}|}$$

of the unit sphere S^2. Let Q_j be the shorter great circle arc connecting a_j and a_{j-1}; their union is a closed curve Q in S^2 (analogous to the unit tangent vector curve $s \mapsto C'(s)$ of a smooth curve C). For convenience, let X be the north pole of S^2, and let the equator E_X be the intersection of the sphere with the xy-plane.





Note that

$$Q_j \text{ crosses } E_X \iff \begin{Bmatrix} a_j \text{ is a relative maximum} \\ \text{or minimum of } P(s) \cdot X \end{Bmatrix}.$$

Consequently, if the equator E_X does not contain a vertex of Q, then

$$2\mu_P(X) = \sum_j \#\{\text{intersections of } Q_j \text{ with } E_X\} = \sum_j \mu_{Q_j}(X), \text{ say.}$$

Since the set $\{X \in S^2 : E_X \text{ contains a vertex of } Q\}$ is of measure 0, it follows that

$$2\int_{S^2} \mu_P \, dA = \sum_j \int_{S^2} \mu_{Q_j} \, dA.$$

To evaluate the integral over S^2 of the function

$$\mu_{Q_j}(X) = \#\{\text{intersections of } Q_j \text{ with } E_X\}$$

we just have to change our point of view slightly. On S^2, the curve Q_j has length α_j, where α_j is the exterior angle at a_j. Consider the two great circles perpendicular to the endpoints of Q_j; they bound a "double lune", as in the figure below. Clearly, Q_j intersects E_X precisely when X is in this double lune, and the area of this region is just

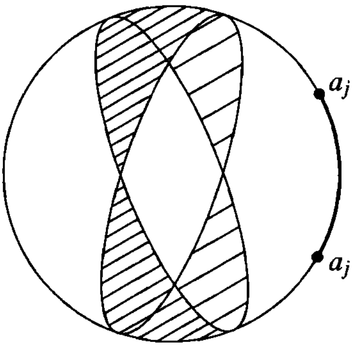
$$\frac{\alpha_j}{\pi} \cdot \text{area } S^2 = \frac{\alpha_j}{\pi} \cdot 4\pi = 4\alpha_j.$$

Consequently,

$$2\int_{S^2} \mu_P \, dA = \sum_j 4\alpha_j,$$

or

$$\int_{S^2} \mu_P \, dA = 2\kappa(P).$$



Finally, for an arbitrary smooth C, we choose a sequence

$$P_1 \prec P_2 \prec P_3 \prec \cdots \to C$$

of polynomial curves, each refining the previous one, which converges to C, and check that for each X,

$$\mu_{P_n}(X) \nearrow \mu_C(X)$$

(a somewhat arduous application of the mean value theorem). It then follows from monotone convergence that

$$\int_{S^2} \mu_{P_n} \, dA \to \int_{S^2} \mu_C \, dA,$$

which yields the desired result.

Milnor's paper actually investigated several other related phenomena, many for curves in $\mathbf{R}^n$; the interested reader is referred to the original paper [1].

2. More about curves

Milnor's next paper on differential geometry [2], coming only a few years after this one, also involved curves in $\mathbf{R}^3$. Most of the results have proofs using methods similar to that of the first paper, and are not of the sort one could easily state in standard differential geometry texts. But there is one result that, to my surprise, does not appear to be mentioned in any book on differential geometry.

Recall that for the unit tangent vector $\mathbf{t}(s)$, we define $\kappa(s) = |\mathbf{t}'(s)|$. If $\kappa(s) \neq 0$, one classically defines the unit *normal* vector $\mathbf{n}(s) = \mathbf{t}'(s)/|\mathbf{t}'(s)|$, and then the *binormal* $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{b}(s)$. The function

$$A(s) = \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} \in \mathrm{SO}(3)$$

must satisfy

$$A'(s) = M(s)A(s)$$

for some skew-symmetric matrix $M(s)$, so that we have the "Serret-Frenet formulas"

$$\begin{aligned} \mathbf{t}' &= \qquad \kappa\mathbf{n} \\ \mathbf{n}' &= -\kappa\mathbf{t} \qquad -\tau\mathbf{b} \\ \mathbf{b}' &= \qquad -\tau\mathbf{n} \end{aligned}$$

where the *torsion* τ is defined by this formula.

For curves with $\kappa \neq 0$ everywhere, the normal curve $\mathbf{n}$ is another closed curve in S^2, whose length is thus

$$\int_0^L \sqrt{\kappa(s)^2 + \tau(s)^2}\, ds.$$

There is a fascinating classical theorem about $\mathbf{n}$:

THEOREM (Jacobi, 1842). If $\mathbf{n}$ is a simple curve on S^2, then it divides S^2 into two parts of equal area.

Milnor added another result:

THEOREM. If $\tau(s) \geq 0$ for all s, but is not identically 0, then the length of $\mathbf{n}$ is $\geq 4\pi$.

3. Other papers on differential geometry

After these two papers on curve theory, Milnor's next excursion into differential geometry [3] almost seems to be following the prescribed undergraduate route, since the main results are concerned with surfaces (here we conveniently ignore the fact that in the interim he had also published his famous paper on manifold homeomorphic to the 7-sphere, etc!). Actually, [3] isn't really about differential geometry *per se*, for it is concerned with connections that do not necessarily arise from metrics.

Moving on to higher dimensions, we come to the brief paper [4] (less than a page, including footnotes). The question had already been raised, to what extent the eigenvalues of the Laplace operator determine the shape of a region ("Can you hear the shape of a drum?"). In [4] Milnor pointed out that for a lattice $L \subset \mathbf{R}^n$, information about the eigenvalues of the Laplace operator on the quotient manifold $\mathbf{R}^n/L$ could be formulated in terms of the dual lattice L^*. From the properties of certain known lattices $L_1, L_2 \subset \mathbf{R}^{16}$ it is then simple to conclude that the Laplace operators on $\mathbf{R}^{16}/L_1$ and $\mathbf{R}^{16}/L_2$ have the same eigenvalues, although the manifolds are not isometric (you can't hear the shape of drums that are topologically 16-dimensional tori).

Here we see the typical Milnor touch, whereby an apparently innocuous fact is employed in an unfamiliar context to yield surprising results. One of the most striking instances of this approach is to be found in the next paper, [5], which has given rise to an entire little subject of its own. A Riemannian manifold $(M, \langle\ ,\ \rangle)$ with curvature tensor R has a "mean curvature tensor" $K(X, Y)$, which plays a role in many classical theorems that can be proved using methods of Morse Theory. For example, if M is compact and $K(X, X) > 0$ for all $X \neq 0$, then $\pi_1(M)$ is finite (Myers) while if M is complete and $K(X, X) \leq 0$ for all X, then $\pi_i(M) = 0$ for $i \geq 2$ (Hadamard-Cartan).

More elementary methods allow one to compare the volume ω_n of the unit disk in $\mathbf{R}^n$ with the volume $V(r)$ of

$$N_r(x_0) = \{y : d(x_0, y) \leq r\}$$

(where d is the distance function on M determined by $\langle\ ,\ \rangle$): If we have $K(X, X) \geq 0$ for all X, then

$$V(r) \leq r^n \omega_n. \tag{1}$$

On the other hand, if $K(X, X) \leq -\alpha^2 < 0$ for all X, then

$$V(r) \geq n\omega_n \int_0^r \left(\frac{\sinh \alpha x}{\alpha}\right)^{n-1} dx;$$

this expression is asymptotic to $2ce^{\lambda r}$ for some c and $\lambda = (n-1)\alpha$, so

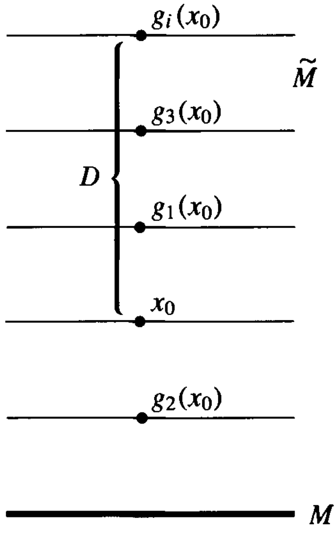
$$V(r) > ce^{\lambda r} \tag{2}$$

for large r.

Milnor observed that the simple relations (1) and (2) have immediate consequences for the fundamental group of M. Suppose first that M is complete, and consider any finitely generated subgroup G of $\pi_1(M)$, with generators $g_1, \ldots, g_p$. We identify elements of G with covering transformations of the universal covering space $\widetilde{M}$ of M. Choose a point $x_0 \in \widetilde{M}$, and let

$$D = \max_i d(x_0, g_i(x_0)),$$

as in the figure below.



We define the *growth function* γ of G (for this particular choice of generators $g_1, \ldots, g_p$) by

$$\gamma(k) = \text{number of different words of length} \leq k.$$

If $g \in G$ is a word of length $\leq k$, then we have $d(x_0, g(x_0)) < k \cdot D$. Consequently,

the neighborhood $N_{k \cdot D}(x_0)$ contains $\gamma(k)$ different points $g(x_0)$, $g \in G$.

If we choose $\varepsilon > 0$ so that $N_\varepsilon(x_0)$ is disjoint from all $g(N_\varepsilon(x_0))$ with $e \neq g \in G$, then

$N_{k \cdot D + \varepsilon}(x_0)$ contains $\gamma(k)$ disjoint sets $g(N_\varepsilon(x_0))$, $g \in G$.

Taking volumes, we have

$$\gamma(k) \cdot V(\varepsilon) \leq V(kD + \varepsilon),$$

hence we can write

$$\gamma(k) \leq CV(kD + \varepsilon) \leq CV(kD + k\varepsilon)$$

for some C. Hence, using (1) we have

THEOREM. If M^n is complete with a positive semidefinite mean curvature tensor, then the growth function γ for any finitely generated subgroup of $\pi_1(M)$ satisfies

$$\gamma(k) \leq \text{constant} \cdot k^n.$$

Once you see this proof it is, of course, hard to believe that you couldn't have done it yourself.

There is a corresponding result for sectional curvatures less than zero:

THEOREM. If M is compact with a negative mean curvature tensor, then the growth function γ of $\pi_1(M)$ is at least exponential:

$$\gamma(k) \geq a^k$$

for some $a > 1$.

The proof of this uses (2) instead of (1). Though similar, it is a bit more complicated, but I wouldn't want to present it here and deprive you of the pleasure of deriving it for yourself.

Two other differential geometric papers, [6], [8], appeared in the *American Mathematical Monthly.* As you might suspect, they are concerned with more elementary problems, but, once again, their results are simultaneously striking and simple.

Finally, [7] is a quite long review article, in which an enormous amount of known material is tied together coherently, with new simplified proofs of much of this material. In this respect it is quite typical of Milnor's expository work, which is just as illuminating as his research. And this brings us, of course, to *Morse Theory* [9], surely the best known of all Milnor's writings on differential geometry.

4. A personal reminiscence

I am glad to have had a small part in the writing of that book, and I certainly learned a great deal during the process. To end on a somewhat more personal note, I would like to recount something of what this learning experience was like.

The book grew out of a series of lectures given by Milnor at Princeton. Robert Wells and I, having volunteered to turn these lectures into written notes, spent a weekend conscientiously setting down the first lecture, which we proudly presented to Milnor. A few days later he returned our notes, saying that they really seemed quite nice, and that there were just a few comments. What this turned out to mean was that there were remarks or queries for only about half of what we had written. And after we rewrote the lecture to account for these problems, it was much improved: now only about a *quarter* of what we had written needed changes. Of course, at the same time we had also written up the second lecture, half of which needed changes. So, as we wrote up the third lecture, we wrote the second draft of the second lecture, only a quarter of which then needed changes, and the third draft of the first lecture, which was really much improved—only about an *eighth* of what we had written needed changes.

Fortunately, this process did eventually converge, and a set of mimeographed notes finally appeared, which was so popular that Milnor was prevailed upon to produce a book for the Annals of Mathematics Studies. I suppose I shouldn't have been surprised when I first saw the manuscript for this book; at its basis were the mimeographed notes, but about half of that had been pasted over with changes, and then about half of these had further changes written over them, ...

And the most exasperating aspect of this constant revision process was this: Each time, the notes really did get better!

During this time I hope that I learned something about writing mathematics books, and I certainly had the opportunity to observe a singular mathematical mind at work. For example, at one point, we had provided proofs for three basic properties of the curvature tensor,

$$R_{ijkl} = -R_{ijlk}$$
$$R_{ijkl} = -R_{jikl}$$
$$R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

As we were standing in the common room at Fine Hall [this was the real Fine Hall, not the modern upstart], the question arose how to give a proof for the fourth property,

$$R_{klij} = R_{ijkl}.$$

You don't have to have any idea just what the curvature tensor is to understand this, because the fourth relation is a formal consequence of the first three. Now the derivation certainly isn't a major affair, but it can become confusing when six or seven graduate students are all trying to do it at once. So, while we were fumbling around, Milnor did what was for him a very typical thing: he simply went off by himself, sat down in a chair at the side of the room, took out a piece of paper, and quietly started writing. And a minute or two later, as we were still flailing, Milnor announced "Oh I see, it all depends on the geometry of the octahedron!" and held forth the diagram that we all know and love on page 54 of *Morse Theory.*

Of course, things like this happened all the time, and after a while they aren't all that surprising. And then you start think, well I know this guy is certainly very bright at mathematics, but is he just as bright about other things? So, I'd like to offer a proof of that also.

One evening in the common room seemed to be an especially unproductive time—no one was playing chess or Blitzkrieg, or any of the other popular time-wasters—and into this languid atmosphere was introduced the idea of writing limericks about various professors. After a few preliminary attempts, indicating that some names wouldn't be very hard to deal with, it soon became apparent that the only truly interesting challenge was posed by Papakyriakopoulos.

As feeble attempts to deal with this interesting case were being proposed, Milnor happened to wander in, and asked what we were doing. And then, as we continued to fumble around, Milnor did a very typical thing: he simply went off by himself, sat down in a chair at the side of the room, took out a piece of paper, and quietly started writing. And a minute or two later, with a shy smile he said "How about this", diffidently offering up a folded piece of paper.

He had already left the room by the time we could unfold the paper, and read the proffered limerick:

The perfidious Lemma of Dehn
Drove mathematicians insane
But Christos Pap
akyriakop
oulos proved it without any pain.

I've always regarded it as one of his finest results.

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