

## ERRATA IN DO CARMO, DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

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This is a list of errata in do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976 (25th printing). The errata were discovered by Bjorn Poonen and some students in his Math 140 class, Spring 2004: Dmitriy Ivanov, Michael Manapat, Gabriel Pretel, Lauren Tompkins, and Po Yee Wong. Some errata were discovered later by Brent Doyle.

- p. 5, line 9, “using properties 3 and 4”: Actually, property 2 also is being used, since property 4 gives linearity in the second variable only.
- p. 5, bottom: The definition of the tangent line is confusing. It is not the line passing through the points  $\alpha(t)$  and  $\alpha'(t)$  but rather the line through  $\alpha(t)$  in the direction of  $\alpha'(t)$ , namely  $\{\alpha(t) + \lambda\alpha'(t) : \lambda \in \mathbb{R}\}$ .
- p. 7, Exercise 3, “On  $0B$  mark off the segment  $0p = CB$ .” It would be better to say “On  $0B$  mark a point  $p$  such that  $0p = CB$ .”
- p. 8, Figure 1-8: The labelling is wrong: the points  $p$  and  $C$  should lie on the same half-line  $r$  through  $0$  as  $B$ .
- p. 8, Figure 1-8: It might be good to add a point labelled  $Y$  on the  $y$ -axis.
- p. 8, Figure 1-9: The angle labelled  $t$  is in the wrong place. (See problem 4 on p. 7.)
- p. 13, line 4: Change “It follows from property 4 that the vector product  $u \wedge v \neq 0$  is normal to a plane generated by  $u$  and  $v$ .” to “It follows from property 4 that if  $u$  and  $v$  are linearly independent, then  $u \wedge v$  is normal to the plane spanned by  $u$  and  $v$ .” or better yet, “It follows from properties 3 and 4 that if  $u$  and  $v$  are linearly independent, then  $u \wedge v$  is a nonzero vector normal to the plane spanned by  $u$  and  $v$ .”
- p. 13, bottom: Again change “a plane generated by” to “the plane spanned by”
- p. 13, bottom: Change “a parallelogram” to “the parallelogram” (twice). Also it would be better to speak of “the parallelogram formed by  $u$  and  $v$ ” instead of “the parallelogram generated by  $u$  and  $v$ ”.
- p. 14, Exercise 5: change “the equation” to “an equation”.
- p. 17, line 1, “Therefore,  $\alpha''(s)$  and the curvature remain invariant under a change of orientation.”: This is not a direct consequence of the previous statement, so replace “Therefore” by “Similarly”.
- p. 18, last sentence: It would be better to change “It follows that” to “Similarly,”
- p. 18, last sentence: change “remains” to “remain”.
- p. 21: In the right part of Figure 1-16, the labels  $e_1$  and  $e_2$  should be reversed.
- pp. 47–48: Exercise 6 is about a *convex* curve, but the curve in Figure 1-37 is not convex.
- p. 49, Exercise 12: The ratio  $M_1/M_2$  actually equals  $1/2$ . In the solution on p. 480, the inner integral in the definition of  $M_1$  should go from  $0$  to  $1/2$ , so  $M_1 = \pi$ .

- p. 57, Figure 2-5: The angle  $\phi$  should be  $\varphi$  to be consistent with the text.
- p. 59, proof of Proposition 2: change “axis” to “axes”.
- p. 59, proof of Proposition 2: change “in  $\mathbb{R}^3$  where  $F$  takes its values” to “in the image of  $F$ ”.
- p. 61, before Example 3: The notion of “connected” given here is usually called “path connected”. (But the definitions are equivalent in the case of regular surfaces, so no harm is done.)
- p. 61, 8th line from bottom: change “by contradiction” to “for sake of obtaining a contradiction”.
- p. 70, Proposition 1: There’s no need to mention  $p$ .
- p. 71, middle, before third display: change “axis” to “axes”.
- p. 72, Definition 1: This requires the notion “ $V$  is open in  $S$ ”, which is not defined until the appendix to Chapter 5. The definition of that notion should appear earlier in the book.
- p. 72, third line after Definition 1:  $p \in \mathbf{x}(V)$  should be  $p \in \mathbf{y}(V)$ .
- p. 76, Example 4: The definition of regular curve does not force  $C$  to be connected, so the assumption that  $C$  does not meet the  $z$ -axis needs to be replaced by the assumption that  $C$  is contained in the open right half of the  $xz$ -plane.
- p. 76, Example 4, first display: It is confusing to include the last inequality  $f(v) > 0$  here, since it is not part of the parametrization, but rather a hypothesis on  $f$ .
- p. 81: Exercise 10 is poorly stated. It is not clear whether  $p$  and  $q$  are in  $C$ . As defined, regular curves are not supposed to have “endpoints”. But if  $p$  and  $q$  are not part of  $C$ , and  $C$  is contained in the open right half-plane, then we get a regular surface of revolution automatically.
- p. 82: Exercise 15b is wrong as stated, even if one assumes  $t_0 = h(\tau_0)$ . For instance, if  $C$  is a circle, then the left and right hand sides could be the lengths of the major and minor arcs connecting two points, respectively. In other words,  $h$  might not be defined on the whole interval  $[\tau_0, \tau]$ , in which case one cannot perform the substitution to transform one integral into the other.
- p. 85, Figure 2-24: change  $\phi$  to  $\varphi$  (twice).
- p. 88, Exercise 3: “Tangent plane” means two different things in the two parts of this problem. In the first part, it is a plane passing through  $p_0$ . In the second part, it is a subspace of  $\mathbb{R}^3$ , that is, a plane passing through the origin.
- p. 94, Example 3: In the parametrization, there is no need to restrict  $u$  to the interval  $(0, 2\pi)$ .
- p. 97, definition of domain: It is not clear whether the boundary is the boundary as a subset of  $\mathbb{R}^3$  or the boundary as a subset of  $S$ . Either way, we run into trouble.

If it is the boundary in  $\mathbb{R}^3$ , then a region in  $S$  need not be contained in  $S$ ! For instance, if we slice an infinite cylinder in two (with a circular cross section),  $S$  could be one of the open halves, so its boundary is the circle, and its closure is a region in  $S$ !

If it is the boundary in  $S$ , then just below Figure 2-28 the assumption that  $R$  is bounded does not imply that  $Q = \mathbf{x}^{-1}(R)$  is bounded, and the integral defining the area could diverge.

Perhaps one should require in the definition of domain that its closure in  $S$  be compact?

- p. 99, Exercise 2: Add a comma between  $\varphi$  and  $\theta$ .
- p. 100, Exercise 8: change “quadratic” to “fundamental”.
- p. 109, Exercise 1: One must assume that  $V_1$  and  $V_2$  are connected.
- p. 118, definition of “neighborhood”: It is more common to define a neighborhood of  $p$  to be any set (open or not) that contains an open set containing  $p$ .
- p. 120, sentence after the second display beginning “In other words,”: This sentence is incorrect and should be deleted.
- pp. 121–122, proof of Proposition 1: The notation  $S_\delta(p)$  should be replaced by  $B_\delta(p)$ , which was used earlier to denote balls. (This appears in three places.)
- p. 122, end of Example 4:  $\mathbb{R}^n$  should be  $\mathbb{R}^m$ .
- p. 123, definition of “continuous in  $A$ ”: It is more common to define this to mean that for all  $a \in A$  and all  $\epsilon > 0$ , there exists  $\delta > 0$  such that the conditions  $x \in A$  and  $|x - a| < \delta$  imply  $|F(x) - F(a)| < \epsilon$ . This definition agrees with continuity with respect to the subspace topology on  $A$ , whereas do Carmo’s definition does not. (To see that the definitions do not agree, consider  $A = \mathbb{R} - \{1/n : n \geq 1\}$ , and define  $F: A \rightarrow \mathbb{R}$  so that  $F(x) = 1/n$  if  $x \in (1/(n+1), 1/n)$  for some integer  $n \geq 1$ , and  $F(x) = 0$  otherwise. This should be continuous on  $A$ , but is not by do Carmo’s definition.)
- p. 124, Proposition 5: The statement should begin “Let  $f: [a, b] \rightarrow \mathbb{R}$ ”.
- p. 124, Proposition 6 (Heine-Borel): The set  $I$  has not been defined. Also, the  $I_\alpha$  need to be open as subsets in  $[a, b]$ , not open intervals in  $\mathbb{R}$  that are contained in  $[a, b]$ . Since this appendix has not defined the notion of one set being open *in another*, it would be best to restate the result as follows:  
 Let  $[a, b]$  be a closed interval, and let  $I_\alpha$ ,  $\alpha \in A$ , be a collection of open intervals such that  $[a, b] \subseteq \bigcup_\alpha I_\alpha$ . Then it is possible to choose a finite number  $I_{k_1}, \dots, I_{k_n}$  of  $I_\alpha$  such that  $[a, b] \subseteq \bigcup_{i=1}^n I_{k_i}$ .
- p. 125, line -2: the word “performed” should perhaps be changed to “taken”.
- p. 127, Definition 1, “we associate a linear map”: It is not proved to be linear until the next Proposition 7. So perhaps it is better to call it just a “map” at this point.
- p. 132, line -3: The second left hand side should have  $(0, 1)$  instead of  $(1, 0)$ .
- p. 136, Definition 1: Delete the | after the semicolon in the display.
- p. 146, 4 lines from the bottom, “any of the sides of the tangent plane”: it would better to replace “any” with “either”.
- p. 147, just after Definition 8: It would make more sense to refer to Example 5 instead of “the method of Example 6”.
- p. 153, second line from the bottom: “parametrization” should be “parametrizations”.
- p. 154, line 4, “all functions to appear below denote their values at the point  $p$ ”: Strictly speaking, some of the functions are functions of  $t$  and should be evaluated at  $t = 0$ .
- p. 154, near the bottom: “coefficients” should be “coefficients”.

- p. 155: It may be preferable to write equation (3) in the form

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

(the transpose of the current version of the equation), so that the matrix  $(a_{ij})$  that appears in it is exactly the matrix of  $dN_p$ , instead of its transpose.

- p. 155, middle, just before the formula for  $a_{11}$ : Each  $x$  in  $\{x_u, x_v\}$  should be bold.
- p. 161, line 2 of Example 4: change “changed” to “replaced”.
- p. 162, middle, “If a parametrization of a regular surface is such that  $F = f = 0$ , then the principal curvatures are given by  $e/E$  and  $g/G$ .” A more enlightening explanation of this is given by the equation following equation (3) on page 155.
- p. 214, first display: Some indices are backwards (the  $ij$ -th entry of a matrix is traditionally the entry in the  $i$ -th row and  $j$ -th column, and in fact this convention is followed on page 154). Change this line to the following:

$$\alpha_{ij} = \langle Ae_j, e_i \rangle = \langle e_j, Ae_i \rangle = \langle Ae_i, e_j \rangle = \alpha_{ji};$$

- p. 215, Lemma: The letter “a” of the word “at” should be italicized.
- p. 221, top right of Figure 4-2:  $d\phi_p(W)$  should be  $d\phi_p(w)$ .
- pp. 224–225: The function  $F$  is being confused with  $F \circ \bar{x}$  (which appears on page 225). For  $F$  to be a function from  $U$  to  $\mathbb{R}^3$ , it should be defined as a function of  $x$  and  $y$ . For  $F \circ \bar{x}$  to make sense,  $F$  should be a function of  $x, y, z$ . But  $F$  was defined as a function of  $\rho$  and  $\theta$ !
- p. 226, display in Definition 3: Since the subscript  $p$  is used on the inner product on the right, it would make sense to use the subscript  $\varphi(p)$  on the left.
- p. 226, second to last display: The range for  $\theta$  should be  $0 \leq \theta \leq \pi$ .
- p. 227, sentence after Proposition 2: With the given definition of local conformality, it seems to be very difficult to prove that it is a symmetric relation, unless one accepts the unproven theorem on page 227. Also, the sentence as written misleadingly suggests that transitivity is the only property required for an equivalence relation.
- p. 232, end first paragraph: Italicize the first appearance of “Christoffel symbols” instead of the second.
- p. 234, sentence after Theorema Egregium: Presumably the codomain of  $\varphi$  was supposed to be a possibly different surface  $\bar{S}$ . In this case, the  $S$  near the end of this sentence also should be  $\bar{S}$ .
- p. 238, sentence before Definition 1: Start this sentence with “The vector field” (to avoid starting it with the mathematical symbol  $w$ ), and change “for every  $p \in U$ ” to “at every  $p \in U$ ”.
- p. 239, middle: “Sec. 4-1” should be “Sec. 4-3”.
- p. 239: In (1), the functions  $a$  and  $b$  depend on the curve  $\alpha$ , so it should be explained that the values of  $a'$  and  $b'$  depend on  $\alpha$  only through its tangent vector.
- p. 243, Figure 4-12:  $\phi$  should be  $\varphi$ .
- p. 245, Figure 4-14:  $\phi$  should be  $\varphi$  (4 times).
- p. 245, last line of Definition 8: “for all  $t \in I$ ” should be “at all  $t \in I$ ” (to match the previous part of Definition 8).
- p. 246, Example 3: “there exists exactly one geodesic  $C \subset S$  passing through  $p$  and tangent to this direction”. Strictly speaking, any connected open neighborhood of

$p$  in  $C$  will be another geodesic, so strictly speaking it is not quite unique. This sloppiness persists throughout the section. For instance:

- p. 246, end of Example 3: Great circles are not the only geodesics on the sphere: arcs of great circles (including the empty arc!) are also geodesics.
- p. 247, last paragraph, first sentence: lines are not the only geodesics of the plane (open intervals in lines are also geodesics)
- p. 260, Exercise 3: Same comment.
- p. 248, Definition 10: change “contained on” to “contained in”.
- p. 248, last sentence of Definition 10: “[ $D\alpha'(s)/ds$ ] =  $k_g$ ” should follow “algebraic value”, not “covariant derivative”.
- p. 249, Figure 4-18: change  $\phi$  to  $\varphi$  in two places.
- p. 249, Figure 4-18: The formula for  $|k_g|$  can be simplified to  $|k_g| = |\cotan \varphi|$ . (No need for the 1!)
- p. 249, middle: There is no need to mention  $k_n$ ; Figure 4-18 shows directly that

$$|k_g| = |k \cos \varphi| = \left| \frac{1}{\sin \varphi} \sin(\pi/2 - \varphi) \right| = |\cotan \varphi|.$$

- p. 249, last line: change “relatively to any” to “relative to either”.
- p. 251, middle, “and observing that  $\langle v, \bar{v} \rangle = 0$ ,  $\langle v, v' \rangle = 0$ ”: After these two equations, “ $\langle \bar{v}, \bar{v}' \rangle = 0$ ” should be inserted, since it also is being used.
- p. 253, Proposition 4: change “the angle that  $\mathbf{x}_u$  makes with  $\alpha'(s)$ ” to “the angle from  $\mathbf{x}_u$  to  $\alpha'(s)$ ”.
- p. 255, Proposition 5: The uniqueness statement should be as follows: For each  $\epsilon > 0$  for which there exists a parametrized geodesic  $\gamma: (-\epsilon, \epsilon) \rightarrow S$  with  $\gamma(0) = p$  and  $\gamma'(0) = w$ , such a parametrized geodesic is unique. Maybe it would be better to state a result for an arbitrary interval  $(a, b)$  with  $a < 0 < b$ . The parametrized geodesic with minimal  $a$  and maximal  $b$  (and  $\gamma(0) = p$  and  $\gamma'(0) = w$ ) is truly unique, and all others (with  $\gamma(0) = p$  and  $\gamma'(0) = w$ ) are restrictions of this one to subintervals of  $(a, b)$  containing 0.
- p. 260, Exercise 1b: “nonrectilinear” is not defined. For this exercise (and for the application of this exercise to Exercise 8), I suggest changing this hypothesis to “ $C$  does not contain an open segment of a straight line”.
- p. 261, Exercise 7(a): The conclusion is false if  $\theta = \pi/2$ .
- p. 261, Exercise 7(b): It’s not clear what “at the points where  $C$  meets their axes” is referring to. I suggest changing it to “at the endpoints of the major and minor axes of the ellipse” (and adding the assumption  $0 < \theta < \pi/2$  so that this makes sense).
- p. 261, Exercise 10: Since the geodesic curvature could have a sign, it should be made clear that “the curvature of the plane curve” refers to the signed curvature defined on page 21, using the orientation of  $T_p(S)$  coming from the orientation of  $S$  that is implicitly used in defining the geodesic curvature. Or else the problem should discuss only the absolute value of the geodesic curvature.
- p. 262, Exercise 17: The first conclusion is false: It can happen that for all  $\epsilon > 0$ , the set  $\mathbf{x}(I \times (-\epsilon, \epsilon))$  fails to be a regular surface. (Consider a curve  $\alpha: (0, 1) \rightarrow \mathbb{R}^3$  such that  $\alpha(s)$  approaches  $(0, 0, 0)$  from the same direction as  $s \rightarrow 0^+$  or  $s \rightarrow 1^-$ , and such that the part of  $\alpha$  near  $s = 0$  is contained in a plane, and the part of  $\alpha$  near  $s = 1$  is contained in a different plane.)

- p. 265, Figure 4-23: Change  $\phi$  to  $\varphi$  (3 times).
- p. 266, paragraph beginning “Assume now...”: One should allow the possibility  $|\theta_i| = 0$ , which is possible even if  $\alpha$  fails to be differentiable ( $C^\infty$ ) at  $t_i$ .
- p. 266, last sentence: This is false;  $\epsilon'$  need not exist. For example, consider the piecewise regular curve in  $\mathbb{R}^2$  given by

$$\alpha(t) := \begin{cases} (-t, 0), & \text{if } t \leq 0 \\ (t, e^{-1/t}(2 + \sin(3/t))), & \text{if } t > 0 \end{cases}$$

The  $y$ -coordinate of  $\alpha'(t)$  changes sign infinitely often as  $t \rightarrow 0^+$ .

Instead one should select  $\theta = \pi$  or  $\theta = -\pi$  as follows. Using a parametrization, reduce to the case where  $\alpha$  is contained in  $\mathbb{R}^2$ , with  $\alpha(t_i) = 0$ , and  $\alpha'(t_i - 0)$  on the negative  $x$ -axis (so  $\alpha'(t_i + 0)$  on the positive  $x$ -axis). For small enough  $\epsilon > 0$ , the trace of  $\alpha$  restricted to  $(t_i - \epsilon, t_i)$  is the graph of a function  $f: (0, \epsilon') \rightarrow \mathbb{R}$  and the trace of  $\alpha$  restricted to  $(t_i, t_i + \epsilon)$  is the graph of a function  $g: (0, \epsilon'') \rightarrow \mathbb{R}$ . Because  $\alpha$  has no self-crossings, the Intermediate Value Theorem implies that either  $f(x) > g(x)$  for all  $x$  for which both are defined, or  $f(x) < g(x)$  for all  $x$  for which both are defined. In the first case, define  $\theta = \pi$ ; in the second, define  $\theta = -\pi$ .

- p. 267, 7 lines from the bottom: It is claimed that a proof of the Theorem of Turning Tangents is contained in Section 5-7, but there the theorem is proved only in the case where  $\alpha$  is a plane curve.
- p. 268, line 3: The  $I$  here (the domain of  $\beta$ ) is different from the  $I$  on the previous page (the domain of  $\alpha$ ).
- p. 271, Figure 4-28: Change  $\phi$  to  $\varphi$ .
- p. 276, statement of Jordan curve theorem before application 1: Insert the word “closed” before “piecewise regular curve”.
- p. 277, application 3: At the end of the first paragraph of the proof, it is claimed that “ $\varphi(\Gamma)$  is the boundary of a simple region of  $P$ ”. This is not necessarily the case, since  $\varphi$  is only a homeomorphism, not a diffeomorphism, and the definition of simple region requires a piecewise  $C^\infty$  boundary.
- p. 277, just below Figure 4-33: The interior of  $\varphi(\Gamma')$  is *not* homeomorphic to a cylinder as claimed. Instead one should apply the global Gauss-Bonnet theorem to the region  $R$  between  $\Gamma$  and  $\Gamma'$  in  $S$ .
- p. 279, line 9: Even if  $K \neq 0$  on  $T$ , it is not necessarily the case that  $\iint_T K d\sigma$  equals the area of  $N(T)$ , because the sign could be wrong, and because the area of  $N(T)$  could be less than expected if  $N$  maps different subregions of  $T$  onto the same set (in this case,  $\iint_T K d\sigma$  counts that part of the area of  $N(T)$  with multiplicity). The statement should be “If  $K \neq 0$  on  $T$  and the restriction  $N|_T$  of  $N$  to  $T$  is injective (or at least these two conditions hold after deleting a subset of measure zero from  $T$ ), then  $|\iint_T K d\sigma|$  equals the area of  $N(T)$ .” Perhaps it would be best to state this for arbitrary regions instead of just triangles, since this is needed to do Exercise 3 on p. 282.
- p. 280, middle of last paragraph: Change “(a ring)” to “(an annulus)”.
- p. 282, last sentence before exercises: Maybe it would be nice to mention that the famous theorem regarding the impossibility of combing a hairy sphere has just been proven.

- p. 282, Exercise 1: One must assume in addition that  $S$  is connected. (Otherwise a disjoint union of two spheres is a counterexample.)
- p. 282, Exercise 3: One must assume that  $S$  has  $K > 0$  everywhere and that  $N$  is injective, so that the area of  $N(A)$  can be computed without worrying about sign or multiplicities.
- p. 282, last line (line 2 of Exercise 5): “ $p \notin C$ ” should be “ $p \in C$ ”.
- p. 282, Exercise 5:  $\varphi$  is being used to denote two different things in this problem (the colatitude, and the angle between two vector fields along  $C$ ).
- p. 425, Definition 1, part 2: There is no need to exclude the case  $W = \emptyset$ ; such an exclusion only clutters the definition. I suggest replacing  
 “For each pair  $\alpha, \beta$  with  $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \phi$ , we have that”  
 with  
 “For each pair  $\alpha, \beta$ , if we set  $W = \mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta)$ , then”.  
 Exactly the same comment applies to Definition 1a on p. 438.
- p. 426, top of Figure 5-46: The subscript on the second  $U$  should be  $\beta$ , not  $p$ .
- p. 426, Remark 1: The statement  
 “This means that any other family satisfying conditions 1 and 2 is already contained in the family  $\{U_\alpha, \mathbf{x}_\alpha\}$ .”  
 neglects the possibility of distinct differentiable structures on the same set. It should be replaced by  
 “This means that the family  $\{U_\alpha, \mathbf{x}_\alpha\}$  is not properly contained in any other family of coordinate charts satisfying conditions 1 and 2 of Definition 1.”
- p. 426, Definition 2: This definition is faulty unless one makes the maximality assumption of Remark 1 on this page. (Without this assumption, it might be impossible to find  $\mathbf{x}$  satisfying  $\mathbf{x}(U) \subset \mathbf{y}(V)$ .)
- p. 427, examples 1 and 2: Change  $\phi$  to  $\emptyset$  (once in each example). Also, it would be easier to read if  $A \circ \mathbf{x}_\alpha(U_\alpha)$  were replaced by  $A(\mathbf{x}_\alpha(U_\alpha))$  (once in each example).
- p. 428, middle of first paragraph: Change  
 “Thus, we must define what is the tangent vector of a curve on an abstract surface.”  
 to  
 “Thus, we must define what the tangent vector of a curve on an abstract surface is.”
- p. 443, Exercise 3: To make sense of the notion of neighborhood of a point in an abstract surface, one needs a notion of open set. I suggest adding the following definition to the text, perhaps after Remark 1 on p. 426: A subset  $V$  of an abstract surface  $S$  is *open* if and only if  $\mathbf{x}_\alpha^{-1}(V)$  is open in  $U_\alpha$  (or equivalently, in  $\mathbb{R}^2$ ) for all  $\alpha$ .
- p. 444, Exercise 5b: “ $d\varphi_p = 0$ ” should be “ $d\varphi_p$  is not an isomorphism”.

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