Thus, we obtain a differentiable map $N\colon \mathbf{x}(U) \to R^3$. We shall see later (Secs. 2-6 and 3-1) that it is not always possible to extend this map differentiably to the whole surface S.

Before leaving this section, we shall make some observations on questions of differentiability.

The definition given for a regular surface requires that the parametrization be of class C^∞, that is, that they possess continuous partial derivatives of *all* orders. For questions in differential geometry we need in general the existence and continuity of the partial derivatives only up to a certain order, which varies with the nature of the problem (very rarely do we need more than four derivatives).

For example, the existence and continuity of the tangent plane depends only on the existence and continuity of the first partial derivatives. It could happen, therefore, that the graph of a function $z = f(x, y)$ admits a tangent plane at every point but is not sufficiently differentiable to satisfy the definition of a regular surface. This occurs in the following example.

Example 3. Consider the graph of the function $z = \sqrt[3]{(x^2 + y^2)^2}$, generated by rotating the curve $z = x^{4/3}$ about the z axis. Since the curve is symmetric relative to the z axis and has a continuous derivative which vanishes at the origin, it is clear that the graph of $z = \sqrt[3]{(x^2 + y^2)^2}$ admits the xy plane as a tangent plane at the origin. However, the partial derivative z_{xx} does not exist at the origin, and the graph considered is not a regular surface as defined above (see Prop. 3 of Sec. 2-2).

We do not intend to get involved with this type of question. The hypothesis C^∞ in the definition was adopted precisely to avoid the study of the minimal conditions of differentiability required in each particular case. These nuances have their place, but they can eventually obscure the geometric nature of the problems treated here.

EXERCISES

***1.** Show that the equation of the tangent plane at (x_0, y_0, z_0) of a regular surface given by $f(x, y, z) = 0$, where 0 is a regular value of f, is

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

2. Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0)$ and show that they are all parallel to the z axis.

3. Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x, y)$, at the point $p_0 = (x_0, y_0)$, is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Recall the definition of the differential df of a function $f\colon R^2 \to R$ and show that the tangent plane is the graph of the differential df_p.

***4.** Show that the tangent planes of a surface given by $z = xf(y/x)$, $x \neq 0$, where f is a differentiable function, all pass through the origin (0, 0, 0).

5. If a coordinate neighborhood of a regular surface can be parametrized in the form

$$\mathbf{x}(u, v) = \alpha_1(u) + \alpha_2(v),$$

where α_1 and α_2 are regular parametrized curves, show that the tangent planes along a fixed coordinate curve of this neighborhood are all parallel to a line.

6. Let $\alpha\colon I \to R^3$ be a regular parametrized curve with everywhere nonzero curvature. Consider the tangent surface of α (Example 5 of Sec. 2-3)

$$\mathbf{x}(t, v) = \alpha(t) + v\alpha'(t), \quad t \in I, v \neq 0.$$

Show that the tangent planes along the curve $\mathbf{x}(\text{const.}, v)$ are all equal.

7. Let $f\colon S \to R$ be given by $f(p) = |p - p_0|^2$, where $p \in S$ and p_0 is a fixed point of R^3 (see Example 1 of Sec. 2-3). Show that $df_p(w) = 2w \cdot (p - p_0)$, $w \in T_p(S)$.

8. Prove that if $L\colon R^3 \to R^3$ is a linear map and $S \subset R^3$ is a regular surface invariant under L, i.e., $L(S) \subset S$, then the restriction $L|S$ is a differentiable map and

$$dL_p(w) = L(w), \quad p \in S, w \in T_p(S).$$

9. Show that the parametrized surface

$$x(u, v) = (v \cos u, v \sin u, au), \quad a \neq 0,$$

is regular. Compute its normal vector $N(u, v)$ and show that along the coordinate line $u = u_0$ the tangent plane of $\mathbf{x}$ rotates about this line in such a way that the tangent of its angle with the z axis is proportional to the inverse of the distance $v(= \sqrt{x^2 + y^2})$ of the point $\mathbf{x}(u_0, v)$ to the z axis.

10. (*Tubular Surfaces.*) Let $\alpha\colon I \to R^3$ be a regular parametrized curve with nonzero curvature everywhere and arc length as parameter. Let

$$\mathbf{x}(s, v) = \alpha(s) + r(n(s) \cos v + b(s) \sin v), \quad r = \text{const.} \neq 0, s \in I,$$

be a parametrized surface (the *tube* of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that, when $\mathbf{x}$ is regular, its unit normal vector is

$$N(s, v) = -(n(s)\cos v + b(s)\sin v).$$

11. Show that the normals to a parametrized surface given by

$$\mathbf{x}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)), \qquad f(u) \neq 0, g' \neq 0,$$

all pass through the z axis.

***12.** Show that each of the equations $(a, b, c \neq 0)$

$$x^2 + y^2 + z^2 = ax,$$
$$x^2 + y^2 + z^2 = by,$$
$$x^2 + y^2 + z^2 = cz$$

define a regular surface and that they all intersect orthogonally.

13. A *critical point* of a differentiable function $f\colon S \to R$ defined on a regular surface S is a point $p \in S$ such that $df_p = 0$.

***a.** Let $f\colon S \to R$ be given by $f(p) = |p - p_0|$, $p \in S$, $p_0 \notin S$ (cf. Exercise 5, Sec. 2-3). Show that $p \in S$ is a critical point of f if and only if the line joining p to p_0 is normal to S at p.

b. Let $h\colon S \to R$ be given by $h(p) = p \cdot v$, where $v \in R^3$ is a unit vector (cf. Example 1, Sec. 2-3). Show that $p \in S$ is a critical point of f if and only if v is a normal vector of S at p.

***14.** Let Q be the union of the three coordinate planes $x = 0$, $y = 0$, $z = 0$. Let $p = (x, y, z) \in R^3 - Q$.

a. Show that the equation in t,

$$\frac{x^2}{a-t} + \frac{y^2}{b-t} + \frac{z^2}{c-t} \equiv f(t) = 1, \qquad a > b > c > 0,$$

has three distinct real roots: t_1, t_2, t_3.

b. Show that for each $p \in R^3 - Q$, the sets given by $f(t_1) - 1 = 0$, $f(t_2) - 1 = 0$, $f(t_3) - 1 = 0$ are regular surfaces passing through p which are pairwise orthogonal.

15. Show that if all normals to a connected surface pass through a fixed point, the surface is contained in a sphere.

16. Let w be a tangent vector to a regular surface S at a point $p \in S$ and let $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(\bar{u}, \bar{v})$ be two parametrizatioos at p. Suppose that the expressions of w in the bases associated to $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(\bar{u}, \bar{v})$ are

$$w = \alpha_1 \mathbf{x}_u + \alpha_2 \mathbf{x}_v$$

and

$$w = \beta_1 \bar{\mathbf{x}}_{\bar{u}} + \beta_2 \mathbf{x}_{\bar{v}}.$$