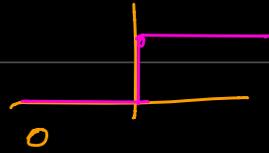Osservazione sulla soluzione del problema di Cauchy

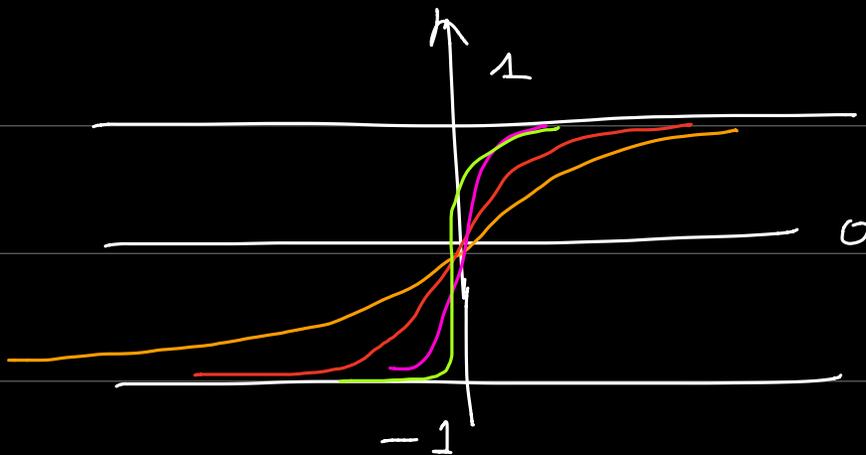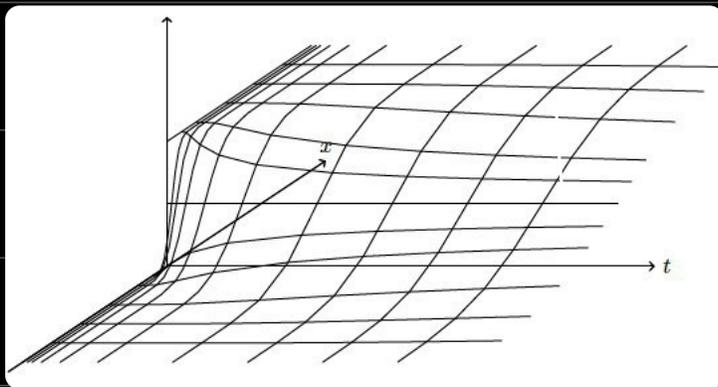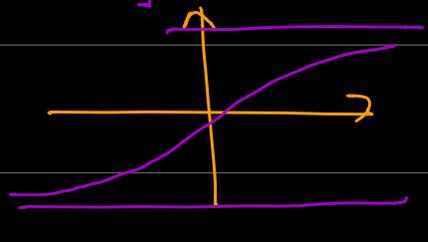$$\begin{cases} u_t = \Delta u \\ u(0,x) = g(x) = \begin{cases} 1 & \text{se } x \geq 0 \\ 0 & \text{se } x < 0 \end{cases} \end{cases}$$

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy = \frac{1}{\sqrt{4\pi t}} \int_0^{+\infty} e^{-\frac{|x-y|^2}{4t}}\, dy$$

$$\frac{x-y}{\sqrt{4t}} = s \implies \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4t}}} e^{-s^2}\, ds = \frac{1}{2}\left(\text{Erf}\left(\frac{x}{\sqrt{4t}} \right) + 1 \right)$$

$$\text{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

ESERCIZI

① Dimostrare che se u risolve

$$\begin{cases} u_t = \Delta u \\ u(0,x) = u_0(x) \end{cases}$$

Allora $v(t,x) = \int_0^x u(t,s)\,ds$ e $w(t,x) = u_x(t,x)$ soddisfano

la stessa equazione con dati iniziali rispettivamente

integrati e derivati. Dobbiamo supporre che questi (e u_0) siano

limitati, in modo da potere utilizzare anche per

questi la formula di rappresentazione.

Scriviamo:

$$u(t,x) = \int_{\mathbb{R}} \Phi(t, x-y)\, u_0(y)\,dy$$

e consideriamo

$$w(t,x) = \int_{\mathbb{R}} \Phi(t, x-y)\, u_0'(y)\,dy$$

Integriamo per parti per $t>0$ e x fissati:

$$\int_{\mathbb{R}} \Phi(t,x-y)\,u_0'(y)\,dy = \lim_{R\to+\infty} \int_{-R}^{R} \Phi(t,x-y)\,u_0'(y)\,dy =$$

$$= \lim_{R\to+\infty} \left\{ \Phi(t,x-y)\,u_0(y) \Big|_{-R}^{R} + \int_{-R}^{R} \Phi_x(t,x-y)\,u_0(y)\,dy \right.$$

$$= \int_{\mathbb{R}} \Phi_x(t,x-y)\,u_0(y)\,dy = \partial_x u(t,x)$$

OSSERVAZIONE IMPORTANTE : abbiamo dimostrato che

$$(\Phi * f)' = \Phi' * f = \Phi * f'$$

é una proprietà fondamentale dei prodotti di convoluzione

Per l'integrale ragioniamo in modo del tutto analogo

— Abbiamo già visto che la soluzione di

$$\begin{cases} u_t = \Delta u \\ u(0,x) = e^{-x^2} = g_0(x) \end{cases}$$

é $u(t,x) = \dfrac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}}$

$Erf(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2} dt$ — Quindi la soluzione di

$$\begin{cases} v_t = \Delta v \\ v(0,x) = Erf(x) = g_1(x) \end{cases}$$

sarà $v_1(t,x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x \dfrac{1}{\sqrt{1+4t}} e^{-\frac{s^2}{1+4t}} ds = \dfrac{2}{\pi} Erf\left(\dfrac{x}{\sqrt{1+4t}}\right)$

— $g_2(x) = x e^{-x^2} = -\dfrac{1}{2} \dfrac{d}{dx} e^{-x^2}$

$\Rightarrow$ la soluzione corrispondente sarà

$$v_2(t,x) = -\dfrac{1}{2} \dfrac{d}{dx}\left(\dfrac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}}\right) = \dfrac{1}{(1+4t)^{3/2}} x e^{-\frac{x^2}{1+4t}}$$

se deriviamo ancora una volta in x otteniamo

$$w(t,x) = -\frac{2}{(1+4t)^{5/2}} x^2 e^{-\frac{x^2}{1+4t}} + \frac{1}{(1+4t)^{3/2}} e^{-\frac{x^2}{1+4t}}$$

che ha come dato iniziale

$$-2x^2 e^{-x^2} + e^{-x^2} = g_3 + g_0$$

Quindi, la soluzione

$$v_3(t,x) = \frac{1}{(1+4t)^{5/2}} x^2 e^{-\frac{x^2}{1+4t}} - \frac{1}{2(1+4t)^{3/2}} e^{-\frac{x^2}{1+4t}} + \frac{1}{2(1+4t)^{1/2}} e^{-\frac{x^2}{1+4t}}$$

(2) Dimostrare, utilizzando la formula di rappresentazione

che

$$\int_{\mathbb{R}} x\, u(t,x)\, dx = \int_{\mathbb{R}} x\, u_0(x)\, dx$$

se $x u_0(x) \in L^1$.

Scriviamo:

$$\int_{\mathbb{R}} x\, u(t,x)\, dx = \int_{\mathbb{R}} x \int_{\mathbb{R}} \Phi(t, x-y)\, u_0(y)\, dy\, dx$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} x\, \Phi(t, x-y)\, u_0(y)\, dy\, dx$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \left[(x-y)\, \Phi(t, x-y)\, u_0(y) + \Phi(t, x-y)\, y\, u_0(y) \right] dy\, dx$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} (x-y)\,\Phi(t,x-y)\,u_0(y)\,dx\,dy + \iint_{\mathbb{R}\times\mathbb{R}} \Phi(t,x-y)\,y\,u_0(y)\,dy\,dx$$

$$= \int_{\mathbb{R}} y\,u_0(y)\,dy$$

Possiamo ragionare in modo analogo per

$$\int_{\mathbb{R}} x^2\,u(t,x)\,dx = \int_{\mathbb{R}\times\mathbb{R}} x^2\,\Phi(t,x-y)\,u_0(y)\,dy\,dx$$

$$x^2 = (x-y)^2 + 2y(x-y) + y^2$$

$$= \iint (x-y)^2\,\Phi(t,x-y)\,u_0(y)\,dx\,dy + \iint 2y(x-y)\,\Phi(t,x-y)\,dy\,dx$$

$$+ \iint \Phi(t,x-y)\,y^2\,u_0(y)\,dx\,dy$$

$$\frac{1}{\sqrt{4\pi t}}\int s^2\,e^{-\frac{s^2}{4t}} \;=\; \frac{4t}{\sqrt{\pi}}\int \underbrace{t^2\,e^{-t^2}}_{t\;te^{-t^2}}\,dt \;=\; \frac{4t}{\sqrt{\pi}}\left[-\frac{1}{2}e^{-t^2}t\,\Big|_{-R}^{R} + \frac{1}{2}\int e^{-t^2}\,dt \right]$$

$$= 2t$$

$$= 2t\int_{\mathbb{R}} u_0(y)\,dy + \int_{\mathbb{R}} y^2\,u_0(y)\,dy$$

$$\int x^2 u_t = \int x^2 u_{xx}\,dx = -\int 2x\,u_x = 2\int u$$

③ Risolvere i problemi di Cauchy

$$\begin{cases} u_t = \Delta u \\ u(0,x) = g(x) \end{cases}$$

dove $g_1(x) = \mathbb{1}_{[-1,1]}$, $g_2(x) = \sin x$, $g_3(x) = e^x$.

— $u(t,x) = \int_{\mathbb{R}} \Phi(t, x-y) g_1(y) dy = \int_{-1}^{1} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy$

pongo $\frac{y-x}{\sqrt{4t}} = s$

$$= \int_{-\frac{1+x}{\sqrt{4t}}}^{\frac{1-x}{\sqrt{4t}}} e^{-s^2} ds = \frac{\sqrt{\pi}}{2} \left[\text{Erf}\left(\frac{1-x}{\sqrt{4t}}\right) + \text{Erf}\left(\frac{1+x}{\sqrt{4t}}\right) \right]$$

— $u(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \sin y \, dy$

pongo $s = \frac{y-x}{\sqrt{4t}}$ $\qquad y = x + \sqrt{4t}\, s$

$$= \frac{1}{\sqrt{\pi}} \int e^{-s^2} \sin(x + \sqrt{4t}\, s) ds =$$

$$= \frac{1}{\sqrt{\pi}} \int e^{-s^2} \left[\sin x \cos(\sqrt{4t}\, s) + \cos x \sin(\sqrt{4t}\, s) \right] ds$$

$$= \frac{1}{\sqrt{\pi}} \left[\sin x \int e^{-s^2} \cos(\sqrt{4t}\, s)\, ds + \cos x \int e^{-s^2} \sin(\sqrt{4t}\, s)\, ds \right.$$

Consideriamo la funzione $t \longmapsto \frac{1}{\sqrt{\pi}} \int e^{-s^2} \cos(\sqrt{4t}\, s)\, ds = g(t)$

derivo rispetto a t. $g'(t) = \frac{-1}{\sqrt{\pi}} \int e^{-s^2} \sin(\sqrt{4t}\, s) \frac{s}{\sqrt{4t}}\, ds$

ora integro per parti in s $\longrightarrow$

$$g'(t) = \frac{-1}{\sqrt{\pi}} \int \left(\frac{d}{ds} e^{-s^2} \right) \sin(\sqrt{4t}\, s) \cdot \frac{1}{\sqrt{4t}}\, dt$$

$$= \frac{-1}{\sqrt{\pi}} \int e^{-s^2} \cos(\sqrt{4t}\, s)\, ds = -g(t)$$

inoltre $g(0) = 1 \implies g(t) = e^{-t}$

$$u(t,x) = e^{-t} \sin x$$

— Formalmente, $u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} e^{y}\, dy$

$(x-y)^2 - 4ty = x^2 - 2xy + y^2 - 4ty = x^2 - 2(x+2t)y + y^2 =$

$(y - (x+2t))^2 - (x+2t)^2 + x^2 = (y - (x+2t))^2 - 4tx - 4t^2$

$$-\frac{(x-y)^2 - 4ty}{4t} = -\frac{(y-(x+2t))^2}{4t} + (x+t)$$

$$\Rightarrow u(t,x) = e^{x+t} \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(y-(x+2t))^2}{4t}} dy = e^{x+t}$$

(4) $Se \quad u(t,x) = v(t) w(x) \Rightarrow u_t - u_{xx} = v'(t) w(x) + v(t) w''(x)$

$se \quad u_t - u_{xx} = 0 \Rightarrow \frac{v'(t)}{v(t)} = -\frac{w''(x)}{w(x)}$

$\Rightarrow \frac{v'(t)}{v(t)} = \alpha = -\frac{w''(x)}{w}$

$se \quad \alpha = 0 \Rightarrow v(t) = const \quad e \quad w'' = 0 \Rightarrow w(x) = ax + b$

$e \quad u(t,x) = a'x + b'$

$se \quad \alpha \neq 0$

$v(t) = A e^{\alpha t} \quad e \quad w(x) = c_1 e^{\sqrt{\alpha} x} + c_2 e^{-\sqrt{\alpha} x}$

$se \quad \alpha > 0 \Rightarrow \alpha = \omega^2 \quad u(t,x) = A e^{\omega^2 t} \left(c_1 e^{\omega x} + c_2 e^{-\omega x}\right)$

$se \quad \alpha < 0 \Rightarrow \alpha = -\omega^2 \quad u(t,x) = A e^{-\omega^2 t} \left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$

(5) $se \quad u(t,x) = \varphi(x - ct) \quad risolve \quad u_t - u_{xx} = 0$

$\Rightarrow \quad c\varphi' + \varphi'' = 0 \Rightarrow (c\varphi + \varphi')' = 0$

$c\varphi + \varphi' = a \Leftrightarrow e^{-cs}(e^{cs}\varphi)' = a \Leftrightarrow e^{cs}\varphi = b + \int_0^1 a e^{ct} dt$

$\Rightarrow \varphi(s) = c_1 e^{ct} + c_2 \qquad \frac{a}{c}(e^{ct} - 1)$