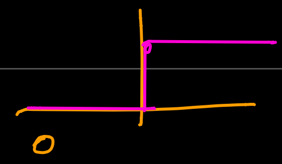
Osservazione sulla soluzione del problema di Cauchy

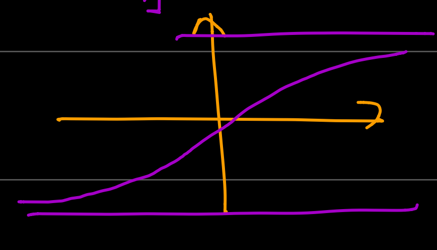
$$\begin{cases} u_t = \Delta u \\ u(0,x) = g(x) = \begin{cases} 1 & \text{se } x \ge 0 \\ 0 & \text{se } x < 0 \end{cases} \end{cases}$$

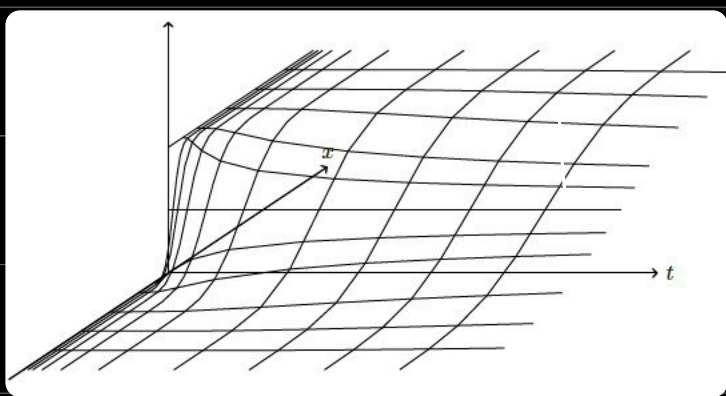


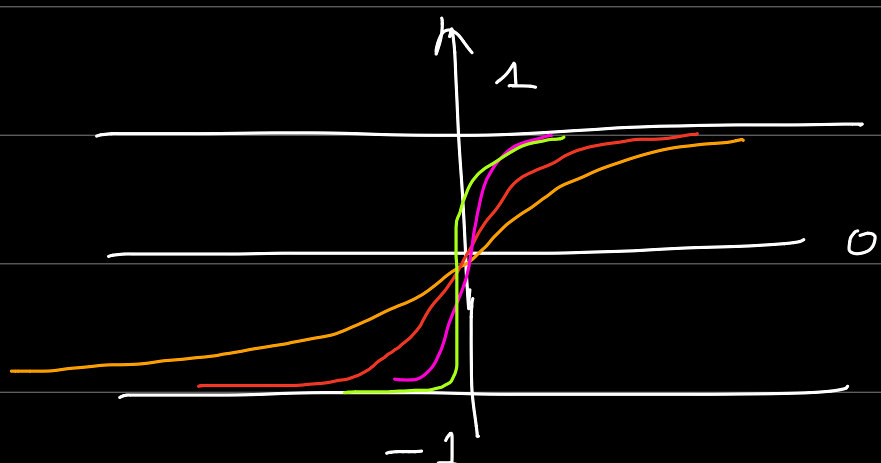
$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy = \frac{1}{\sqrt{4\pi t}} \int_0^{+\infty} e^{-\frac{|x-y|^2}{4t}}\,dy$$

$$\frac{x-y}{\sqrt{4t}} = s \quad \Rightarrow \quad \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4t}}} e^{-s^2}\,ds = \frac{1}{2}\left(\mathrm{Erf}\left(\frac{x}{\sqrt{4t}}\right) + 1\right)$$

$$\boxed{\mathrm{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt}$$







ESERCIZI

① Dimostrare che se u risolve

$$\begin{cases} u_t = \Delta u & \mathbb{R}^+ \times \mathbb{R} \\ u(0,x) = u_0(x) & \mathbb{R} \end{cases}$$

Allora $\boxed{v(t,x) = \int_0^x u(t,s)\,ds}$ e $\boxed{v(t,x) = u_x(t,x)}$ soddisfano la stessa equazione con dati iniziali rispettivamente integrati e derivati. Dobbiamo supporre che questi (e u_0) siano limitati, in modo da potere utilizzare anche per questi la formula di rappresentazione.

Scriviamo:

$$u(t,x) = \int_{\mathbb{R}} \Phi(t,x-y)u_0(y)\,dy \Rightarrow \partial_x u = \int_{\mathbb{R}} \partial_x \Phi(t,x-y)u_0(y)\,dy$$

e consideriamo

$$w(t,x) = \int_{\mathbb{R}} \Phi(t,x-y)\underbrace{u_0'(y)dy}$$

Integriamo per parti per $t>0$ e x fissati:

$$\int_{\mathbb{R}} \Phi(t,x-y)u_0'(y)\,dy = \lim_{R\to+\infty} \int_{-R}^{R} \Phi(t,x-y)u'(y)\,dy =$$

$$= \lim_{R\to+\infty} \left\{ \Phi(t,x-y)u_0(y)\Big|_{-R}^{R} + \int_{-R}^{R} \Phi_x(t,x-y)u_0(y)\,dy \right.$$

$$= \int_{\mathbb{R}} \Phi_x(t,x-y)\,u_0(y)\,dy = \partial_x u(t,x)$$

OSSERVAZIONE IMPORTANTE: abbiamo dimostrato che

$$(\Phi * f)' = \Phi' * f = \Phi * f'$$

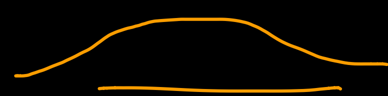
é una proprietà fondamentale dei prodotti di convoluzione

Per l'integrale ragioniamo in modo del tutto analogo

- Abbiamo già visto che la soluzione di

$$\begin{cases} u_t = \Delta u \\ u(0,x) = e^{-x^2} = g_0(x) \end{cases}$$

é $u(t,x) = \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}}$



$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ _ Quindi la soluzione di

$$\begin{cases} v_t = \Delta v \\ v(0,x) = \mathrm{Erf}(x) = g_1(x) \end{cases}$$

sarà $v_1(t,x) = \frac{2}{\pi} \int_0^x \frac{1}{\sqrt{1+4t}} e^{-\frac{s^2}{1+4t}} ds = \frac{2}{\pi} \mathrm{Erf}\left(\frac{x}{\sqrt{1+4t}}\right)$

- $g_2(x) = x e^{-x^2} = -\frac{1}{2} \frac{d}{dx} e^{-x^2}$

$\Rightarrow$ la soluzione corrispondent sarà

$$v_2(t,x) = -\frac{1}{2} \frac{d}{dx}\left(\frac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}}\right) = \frac{1}{(1+4t)^{3/2}} x e^{-\frac{x^2}{1+4t}}$$

se deriviamo ancora una volta in x otteniamo

$$w(t,x) = -\frac{2}{(1+4t)^{5/2}} x^2 e^{-\frac{x^2}{1+4t}} + \frac{1}{(1+4t)^{3/2}} e^{-\frac{x^2}{1+4t}}$$

che ha come dato iniziale

$$-2x^2 e^{-x^2} + e^{-x^2} = g_3 + g_0$$

Quindi, la soluzione

$$v_3(t,x) = \frac{1}{(1+4t)^{5/2}} x^2 e^{-\frac{x^2}{1+4t}} - \frac{1}{2(1+4t)^{3/2}} e^{-\frac{x^2}{1+4t}} + \frac{1}{2(1+4t)^{1/2}} e^{-\frac{x^2}{1+4t}}$$

(2) Dimostrare, utilizzando la formula di rappresentazione

che $$\int_{\mathbb{R}} x\, u(t,x)\,dx = \int_{\mathbb{R}} x\, u_0(x)\,dx$$

se $x u_0(x) \in L^1$.

Scriviamo:

$$\int_{\mathbb{R}} x\, u(t,x)\,dx = \int_{\mathbb{R}} x \int_{\mathbb{R}} \Phi(t,x-y) u_0(y)\,dy\,dx$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} x\, \Phi(t,x-y) u_0(y)\,dy\,dx$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \left[(x-y)\,\Phi(t,x-y) u_0(y) + \Phi(t,x-y)\, y\, u_0(y)\right] dy\,dx$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} (x-y)\,\Phi(t,x-y)\,u_0(y)\,dx\,dy + \iint_{\mathbb{R}\times\mathbb{R}} \Phi(t,x-y)\quad u_0(y)\,dy\,dx$$

$$= \int_{\mathbb{R}} y\,u_0(y)\,dy \qquad \int_{\mathbb{R}} \Phi(t,x)\,dx = 1\ ;\ \int_{\mathbb{R}} x\,\Phi(t,x)\,dx = 0$$

pari in x

possiamo applicare il teorema di Fubini-Tonelli

Possiamo ragionare in modo analogo per

$$\int_{\mathbb{R}} x^2 u(t,x)\,dx = \int_{\mathbb{R}\times\mathbb{R}} x^2\,\Phi(t,x-y)\,u_0(y)\quad dx$$

$$x^2 = (x-y)^2 + 2y(x-y) + y^2$$

$$= \iint (x-y)^2\,\Phi(t,x-y)\,u_0(\ )\,dx\,dy + \iint 2y(x-y)\,\Phi(t,x-y)\,u_0(y)\,dy\,dx$$

$$+ \iint \Phi(t,x-y)\,y^2 u_0(y)\,dx\,dy$$

$$\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{+\infty} s^2 e^{-\frac{s^2}{4t}}\,ds = \frac{4t}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \underbrace{t^2 e^{-t^2}}_{t\,te^{-t^2}}\,dt = \frac{4t}{\sqrt{\pi}}\left[-\frac{1}{2}e^{-t^2}t\Big|_{-R}^{R} + \frac{1}{2}\int e^{-t^2}dt\right]$$

$$te^{-t^2} \Rightarrow -\frac{1}{2}\frac{d}{dt}e^{-t^2}$$

$$= 2t$$

$$= 2t\int_{\mathbb{R}} u_0(y)\,dy + \int_{\mathbb{R}} y^2 u_0(y)\,dy$$

$$\frac{d}{dt}\int x^2 u(t,x) = \int_{-\infty}^{+\infty} x^2 \underset{u_{xx}}{u_t} = \int_{-\infty}^{+\infty} x^2 u_{xx}\,dx = -\int_{-\infty} 2x\,u_x = 2\int_{-\infty} u_0(y)\,dy$$

per parti $[-R,R]$

$R \to +\infty$

③ Risolvere i problemi di Cauchy

$$\begin{cases} u_t = \Delta u \\ u(0,x) = g(x) \end{cases}$$

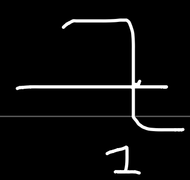
dove $g_1(x) = \mathbb{1}_{[-1,1]}$, $g_2(x) = \sin x$, $g_3(x) = e^x$.

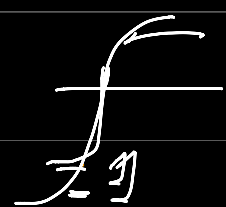
1. $u(t,x) = \int_{\mathbb{R}} \Phi(t, x-y)\, g_1(y)\, dy = \int_{-1}^{1} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy$

pongo $\frac{y-x}{\sqrt{4t}} = s$

$\mathrm{Erf}(-s) = -\mathrm{Erf}(s)$

$$= \frac{1}{\sqrt{\pi}} \int_{-\frac{1+x}{\sqrt{4t}}}^{\frac{1-x}{\sqrt{4t}}} e^{-s^2} ds = \frac{1}{2}\left[\mathrm{Erf}\left(\frac{1-x}{\sqrt{4t}}\right) + \mathrm{Erf}\left(\frac{1+x}{\sqrt{4t}}\right)\right]$$





2. $u(t,x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \sin y \, dy$

pongo $s = \frac{y-x}{\sqrt{4t}}$ $\quad y = x + \sqrt{4t}\, s$

$$= \frac{1}{\sqrt{\pi}} \int e^{-s^2} \sin\left(x + \sqrt{4t}\, s\right) ds =$$

$$= \frac{1}{\sqrt{\pi}} \int e^{-s^2} \left[\sin x \cos\left(\sqrt{4t}\, s\right) + \cos x \sin\left(\sqrt{4t}\, s\right)\right] ds$$

$$= \frac{1}{\sqrt{\pi}}\Big[\sin x \underbrace{\int e^{-s^2}\cos(\sqrt{4t}\,s)\,ds}_{v_1(t)} + \cos x \underbrace{\int e^{-s^2}\sin(\sqrt{4t}\,s)\,ds}_{v_2(t)\equiv 0}\Big]$$

Consideriamo la funzione $\boxed{t \mapsto \frac{1}{\sqrt{\pi}}\int e^{-s^2}\cos(\sqrt{4t}\,s)\,ds} = g(t)$

derivo rispetto a t. $g'(t) = \frac{-1}{\sqrt{\pi}}\int e^{-s^2}\sin(\sqrt{4t}\,s)\,\frac{2s}{\sqrt{4t}}\cdot ds$ $\quad \frac{d}{dt}\left(\sqrt{4t}\,s\right)$

ora integro per parti in s $\rightarrow$

$$g'(t) = \frac{-1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(\frac{d}{ds}e^{-s^2}\right)\sin(\sqrt{4t}\,s)\cdot\frac{1}{\sqrt{4t}}\,dt$$

$$= \frac{-1}{\sqrt{\pi}}\int e^{-s^2}\cos(\sqrt{4t}\,s)\,ds = -g(t)$$

inoltre $g(0) = 1 \implies g(t) = e^{-t}$

$$u(t,x) = e^{-t}\sin x$$

– Formalmente, $u(t,x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} \underbrace{e^{-\frac{(x-y)^2}{4t}}\, e^{y}}_{\text{Converge } \forall t>0 \text{ fissato}}\,dy$

$$(x-y)^2 - 4ty = x^2 - 2xy + y^2 - 4ty = x^2 - 2(x+2t)y + y^2 =$$

$$(y-(x+2t))^2 - (x+2t)^2 + x^2 = (y-(x+2t))^2 - 4tx - 4t^2$$

$$-\frac{(x-y)^2-4ty}{4t} = -\frac{(y-(x+2t))^2}{4t} + (x+t)$$

$$\Rightarrow u(t,x) = e^{x+t} \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(y-(x+2t))^2}{4t}} dy = e^{x+t}$$

(4) Se $u(t,x) = v(t)w(x) \Rightarrow u_t - u_{xx} = v'(t)w(x) + v(t)w''(x)$

se $u_t - u_{xx} = 0 \Longrightarrow \frac{v'(t)}{v(t)} = -\frac{w''(x)}{w(x)}$

$$\Rightarrow \frac{v'(t)}{v(t)} = \alpha = -\frac{w''(x)}{w}$$

se $\alpha = 0 \Rightarrow v(t) = \text{cost}$ e $w'' = 0 \Rightarrow w(x) = ax+b$
e $u(t,x) = a'x+b'$ eon

se $\alpha \neq 0$

$v(t) = A e^{\alpha t}$ e $w(x) = c_1 e^{\sqrt{\alpha}\, x} + c_2 e^{-\sqrt{\alpha}\, x}$

se $\alpha > 0 \Rightarrow \alpha = \omega^2$ $u(t,x) = A e^{\omega t}(c_1 e^{\omega x} + c_2 e^{-\omega x})$

se $\alpha <$ $\Rightarrow \alpha = -\omega^2$ $u(t,x) = A e^{-\omega^2 t}(c_1 \cos(\omega x) + c_2 \sin(\omega x))$

(5) se $u(t,x) = \varphi(x-ct)$ risolve $u_t - u_{xx} = 0$

$$\Rightarrow c\varphi' + \varphi'' = 0 \Rightarrow (c\varphi + \varphi')' = 0$$

$c\varphi + \varphi' = a \Leftrightarrow e^{-cs}(e^{cs}\varphi)' = a \Longleftrightarrow e^{cs}\varphi = b + \int_0^s a e^{ct} dt$

$\Rightarrow \varphi(s) = c_1 e^{-cs} + c_2$ $\qquad \frac{a}{c}(e^{ct}-1)$