The isoperimetric inequality

Let Ω be a bounded domain in $\mathbb{R}^n$. If Ω is smooth enough, its boundary is an $(n-1)$-dimensional manifold and its $(n-1)$-dimensional area is well defined [1]. We aim to discuss the isoperimetric inequality in $\mathbb{R}^n$:

Theorem 0.1 (isoperimetric inequality)**.** *For every bounded domain Ω in $\mathbb{R}^n$ of class C^∞ one has that*

$$\frac{|\partial\Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial B|^n}{|B|^{n-1}} = n^n\omega_n \tag{1}$$

where $|\partial\Omega|$ is the $(n-1)$-dimensional area of $\partial\Omega$, $|\Omega|$ is the Lebesgue measure of Ω, B is any ball in $\mathbb{R}^n$ and ω_n is the n-dimensional volume of the unit ball. Moreover equality holds if and only if Ω is a ball.

This inequality has a long history and admits many proofs. Here we present the Cabré' proof (Butl. Soc. Catalana Mat. **15** (2000), 7–27; see also: Discrete Continuous Dyn. Syst. Ser. A **20** (2008), 425–457). It uses simple ideas from elementary calculus and partial differential equations, in particular certain Neumann problems. It is interesting because one can visualize it in a geometric way and, in that respect, one can recognize the idea of the Alexandrov's method of moving planes.

0.1 Auxiliary Neumann problems

Here we study the following Neumann problems:

$$\begin{cases} \Delta u = c & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

where Ω is a bounded regular domain in $\mathbb{R}^n$ and $\frac{\partial u}{\partial \nu}$ denotes the outer normal derivative of u at a point of $\partial\Omega$ defined, for every $x \in \partial\Omega$, by

$$\frac{\partial u}{\partial \nu}(x) = \lim_{t\to 0^-} \frac{u(x+t\nu) - u(x)}{t}$$

where $\nu = \nu(x)$ is the outer unit versor at x. In particular, if $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu\,.$$

Our goal is to show the following result:

[1] One can calculate it on parameterized portions of $\partial\Omega$, proving that it is independent of choice of parameterization. In particular, the (m-dimensional) area of the image of a C^1 mapping $f\colon D \subset \mathbb{R}^m \to \mathbb{R}^n$ (with $m < n$) is given by $\text{area}(f(D)) = \int_D \sqrt{\det(Df(x)^\top Df(x))}\,dx$.

Theorem 0.2. *Assume Ω of class $C^{2,\alpha}$ (with $0 < \alpha < 1$). Problem (2) admits a classical solution $u \in C^1(\overline{\Omega}) \cap C^2(\omega)$ if and only if*

$$c = \frac{|\partial\Omega|}{|\Omega|}\,. \tag{3}$$

Moreover, in this case, the solution belongs to $C^2(\overline{\Omega})$ and is unique up to an additive constant.

Proof. Let us show that (3) is a necessary condition. Let $u \in C^1(\overline{\Omega}) \cap C^2(\omega)$ be a solution of (2). If we integrate the equation, using the divergence theorem, we find

$$0 = \int_\Omega (\Delta u - c) = \int_{\partial\Omega} \nabla u \cdot \nu - c|\Omega| = |\partial\Omega| - c|\Omega|$$

that is (3). Now let us discuss the uniqueness. Let $u_1, u_2 \in C^1(\overline{\Omega}) \cap C^2(\omega)$ be two solutions of (2). Then $u := u_1 - u_2$ solves

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying the equation by u and integrating, by the divergence theorem, we find

$$0 = \int_\Omega u\,\Delta u = \int_{\partial\Omega} u\,\nabla u \cdot \nu - \int_\Omega |\nabla u|^2 = -\int_\Omega |\nabla u|^2\,.$$

Then $|\nabla u| = 0$ in Ω and, since Ω is connected, u is constant, namely u_1 and u_2 differ by an additive constant. Now assume (3) and let us prove that (2) admits a solution. Recall the following result (see: Gilbarg, Trudinger: Elliptic Partial Differential Equations of Second Order, §14.6):

Lemma 0.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ of class C^2. Then there exists $\varepsilon_0 > 0$ such that the mapping*

$$\mathrm{d}_\Omega(x) := \begin{cases} -\mathrm{dist}(x, \partial\Omega) & \text{for } x \in \Omega \\ \mathrm{dist}(x, \partial\Omega) & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

is of class C^2 in $\Sigma = \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, \partial\Omega) < \varepsilon_0\}$. Moreover for every $x \in \Sigma$ there exists a unique $p(x) \in \partial\Omega$ such that $\mathrm{dist}(x, \partial\Omega) = |p(x) - x|$ and $\nabla \mathrm{d}_\Omega(x) = \nu(p(x))$. In addition, if $\partial\Omega \in C^k$, with $k \geq 2$, then $\mathrm{d}_\Omega \in C^k(\Sigma)$.

Note that

$$\frac{\partial \mathrm{d}_\Omega}{\partial \nu}(x) = 1 \quad \forall x \in \partial\Omega\,.$$

With a simple extension argument, one can determine a C^2 function $w\colon \mathbb{R}^n \to \mathbb{R}$ such that $w = \mathrm{d}_\Omega$ in a neighborhood of $\partial\Omega$. Then one solves the Neumann problem

$$\begin{cases} \Delta v + f = 0 & \text{in } \Omega \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

where $f = \Delta w - c$. Assuming that such problem admits a solution v, one immediately checks that the mapping $u = v + w$ solves (2). The rest of this section concerns the study of problem (4).

Lemma 0.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ of class $C^{2,\alpha}$ and let $f \in C^{0,\alpha}(\overline{\Omega})$. If*

$$\int_\Omega f = 0$$

then (4) admits a solution $u \in C^{2,\alpha}(\overline{\Omega})$.

We apply Lemma 0.1.3 with $f = \Delta w - c$. This is possible, because

$$\int_\Omega f = \int_\Omega [\operatorname{div}(\nabla w) - c] = \int_{\partial\Omega} \nabla w \cdot \nu - c|\Omega| = \int_{\partial\Omega} \frac{\partial \mathrm{d}_\Omega}{\partial \nu} - |\partial\Omega| = 0\,.$$

Then the existence of a solution v of problem (4) is guaranteed by Lemma 0.1.3 and the proof of Theorem 0.1.1 is complete. □

Proof of Lemma 0.1.3. For every integrable function $\varphi\colon \Omega \to \mathbb{R}$ denote

$$[\varphi] := \fint_\Omega \varphi = \frac{1}{|\Omega|}\int_\Omega \varphi$$

the average of φ in Ω. Let us introduce the space of mappings in $W^{1,2}(\Omega)$ with null avarage:

$$W^{1,2}_* := \{u \in W^{1,2}(\Omega) \mid [u] = 0\}$$

and the functional

$$F(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right).$$

We show that

$$\exists u \in W^{1,2}_* \quad : \quad F(u) = \inf_{W^{1,2}_*} F\,. \tag{5}$$

Observe that $W^{1,2}_*$ is a closed subspace of $W^{1,2}(\Omega)$. In particular it is a closed and convex subset of a reflexive Banach space. If we show that F is convex and coercive we can conclude, thanks to the generalized Weierstrass theorem. The functional

$$F_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$$

is strictly convex because of the convexity of the mapping $t \mapsto t^2$. The functional

$$F_1(u) = -\int_\Omega fu$$

is convex because it is linear. Then $F = F_0 + F_1$ is strictly convex. Let us show that F is coercive. To this aim we use the following version of the Poincaré inequality:

Lemma 0.5 (Poincaré inequality)**.** *Let $p \in (1,\infty)$. Then there exists a constant $C_\Omega > 0$ such that $\int_\Omega |u - [u]|^p \le C_\Omega \int_\Omega |\nabla u|^p \quad \forall u \in W^{1,p}(\Omega)$.*

Using the Cauchy inequality and the Poincaré inequality, we have that

$$F(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \|f\|_2\|u\|_2 \geq \frac{1}{2}\|\nabla u\|_2^2 - C_\Omega\|f\|_2\|\nabla u\|_2 \quad \forall u \in W^{1,2}_* .$$

If $(u_k) \subset W^{1,2}_*(\Omega)$ is such that $\|u_k\|_{W^{1,2}} \to \infty$, by the Poincaré inequality, $\|\nabla u_k\|_2 \to \infty$ and then, by the previous estimate, $F(u_k) \to \infty$. This means that F is coercive. Hence we can apply the Weierstrass theorem for convex coercive funtionals in convex closed subsets of a reflexive space, obtaining (5). Since u is a minimum point of F in $W^{1,2}_*$ one has that

$$0 = \lim_{t\to 0} \frac{F(u+th) - F(u)}{t} = \frac{\partial F}{\partial h}(u) = \int_\Omega (\nabla u \cdot \nabla h - fh) \quad \forall h \in W^{1,2}_*(\Omega) .$$

In fact, the same holds also for $h \in W^{1,2}(\Omega)$. Indeed, if $h \in W^{1,2}(\Omega)$ we consider $h_0 = h - [h]$. Since $h_0 \in W^{1,2}_*$ we have

$$\int_\Omega (\nabla u \cdot \nabla h - fh) = \int_\Omega (\nabla u \cdot \nabla h_0 - fh_0) - [h]\int_\Omega f = 0$$

because $[f] = 0$. By the Schauder regularity theory[2] , since $f \in C^{0,\alpha}(\overline{\Omega})$ and Ω is of class $C^{2,\alpha}$, one has that $u \in C^{2,\alpha}(\overline{\Omega})$. Hence, by the divergence theorem,

$$0 = -\int_\Omega (\nabla u \cdot \nabla h - fh) = \int_\Omega (\Delta u + f)\, h \quad \forall h \in C^\infty_c(\Omega) .$$

Therefore u solves $\Delta u + f = 0$ in Ω. Moreover, since

$$0 = \int_\Omega (\nabla u \cdot \nabla h - f\, h) = \int_{\partial\Omega} h\, \frac{\partial u}{\partial \nu} - \int_\Omega (\Delta u + f)\, h = \int_{\partial\Omega} h\, \frac{\partial u}{\partial \nu} \quad \forall h \in C^\infty_c(\Omega)$$

we infer that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. In conclusion, u solves (4). □

0.2 Proof of Theorem 0.0.1

Fix a bounded domain Ω in $\mathbb{R}^n$, of class $C^{2,\alpha}$ and let $u \in C^2(\overline{\Omega})$ be a solution of the Neumann problem (2), given by Theorem 0.1.1.

Definition 0.6. *The **lower contact set** of u is the set defined by*

$$\Gamma_u = \{x \in \Omega \mid u(y) \geq u(x) + \nabla u(x)\cdot(y-x)\ \forall y \in \Omega\}.$$

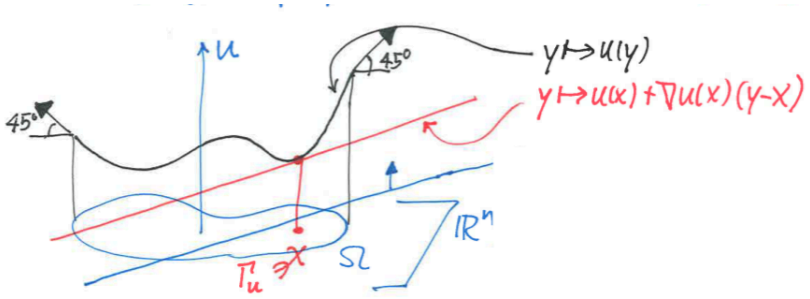
[2] **Theorem:** Let $\alpha \in (0,1)$, Ω bounded domain in $\mathbb{R}^n$ of class $C^{2,\alpha}$, $f \in C^{0,\alpha}(\overline{\Omega})$ and $g \in C^{1,\alpha}(\overline{\Omega})$ be such that $\int_\Omega f = \int_{\partial\Omega} g$. Then the Neumann problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$

admits a solution (unique, up to an additive constant) $u \in C^{2,\alpha}(\overline{\Omega})$. Moreover

$$\|u - [u]\|_{C^{2,\alpha}} \leq C(\|f\|_{C^{0,\alpha}} + \|g\|_{C^{1,\alpha}})$$

with $C = C(n,\alpha,\Omega)$ (See: G. Nardi, Schauder estimate for solutions of Poisson's equation with Neumann boundary condition, L'Enseignement Mathématique (2) **60** (2014), pp. 423–437).



In other words, the lower contact set is the set of all points in Ω such that the graph of u lies entirely above the tangent plane at the point $(x, u(x))$. The set Γ_u is made by convexity points of u. In fact, we have:

Lemma 0.7. *If $u \in C^2(\overline{\Omega})$, then for every $x \in \Gamma_u$ the Hessian matrix $D^2u(x)$ is positive semidefinite, i.e.*

$$D^2u(x)\xi \cdot \xi \geq 0 \ \forall \xi \in \mathbb{R}^n. \tag{6}$$

Proof. One has

$$u(x + t\xi) = u(x) + t\nabla u(x) \cdot \xi + \frac{t^2}{2} D^2u(x)\xi \cdot \xi + o(t^2) \quad \text{as } t \to 0$$

and, since $x \in \Gamma(u)$ it follows that $t^2 D^2u(x)\xi \cdot \xi + o(t^2) \geq 0$ and (6) follows on dividing by t^2 and letting $t \to 0$. □

Lemma 0.8. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and let u be a solution of (2). Then $\forall p \in \mathbb{R}^n$ with $|p| < 1$ $\exists x \in \Gamma_u$ such that $\nabla u(x) = p$, namely, $B_1 \subset \nabla u(\Gamma_u)$ where B_1 denotes the open unit ball in $\mathbb{R}^n$.*

Proof. Let us consider a hyperplane in $\mathbb{R}^{n+1}$ defined as the graph of the affine map $\mathbb{R}^n \ni x \mapsto p \cdot x + t \in \mathbb{R}$ with fixed $p \in \mathbb{R}^n$, $|p| < 1$, and $t \in \mathbb{R}$ such that this hyperplane stays below the graph of u. Then we increase t, i.e., we move the hyperplane parallel to itself until it touches the graph of u. This occurs in correspondence of the value

$$\sup\{t \in \mathbb{R} \ : \ t \leq u(x) - p \cdot x \ \forall x \in \overline{\Omega}\} = \min\{u(x) - p \cdot x \ : \ x \in \overline{\Omega}\}$$

which is attained at some $x_0 \in \overline{\Omega}$. Assume that the first point of contact with the graph of u, goven by $(x_0, u(x_0))$ occurs with $x_0 \in \partial\Omega$. We want to show that this is impossible, the reason being the fact that $|p| < 1$ and the graph of u arrives at the boundary with an angle of 45° because of the boundary Neumann condition (indeed, considering $\varphi \to (-\varepsilon, 0] \to \mathbb{R}$ given by $\varphi(t) = u(x_0 + t\nu)$ with $x_0 \in \partial\Omega$, one has that $\varphi'(0) = \frac{\partial u}{\partial \nu}(x_0) = 1$). To check that $x_0 \notin \partial\Omega$ we observe that x_0 satisfies

$$\min_{x \in \overline{\Omega}}(u(x) - p \cdot x) = u(x_0) - p \cdot x_0.$$

(Note that, up to a sign, the map $p \mapsto x_0$ is nothing but the Legendre transformation of u). Hence, the function $g(x) = u(x) - u(x_0) - p \cdot (x - x_0)$ is non-negative in $\overline{\Omega}$ and attains its minimum at x_0 on the boundary of Ω. Thus, if ν is the unit outer normal on $\partial\Omega$ at x_0 we have

$$\frac{\partial g}{\partial \nu}(x_0) = \lim_{t \to 0^-} \frac{g(x_0 + t\nu) - g(x_0)}{t} \leq 0.$$

Since $\frac{\partial g}{\partial \nu}(x_0) = \frac{\partial u}{\partial \nu}(x_0) - p \cdot \nu$ and $\frac{\partial u}{\partial \nu} = 1$ on $\partial\Omega$, we obtain

$$p \cdot \nu \geq 1.$$

In particular $|p| \geq 1$. Therefore we proved that if $|p| < 1$, then the first point of contact between the hyperplane graph of $y = x \mapsto p \cdot x + t$ with the graph of u is a point $(x_0, u(x_0))$ with $x_0 \in \Omega$. In this case, since x_0 is a minimum point of $g(x)$ in Ω, it must be $\nabla g(x_0) = 0$. Hence $x_0 \in \Gamma_u$ and $p = \nabla u(x_0)$. □

In view of Lemma 0.2.3, we have

$$\omega_n = |B_1(0)| \leq |\nabla u(\Gamma_u)| = \int_{\nabla u(\Gamma_u)} dp.$$

Now we want to change variable, by setting $p = \nabla u(x)$, with $x \in \Gamma_u$. In general, if $F \colon A \to B$ is a diffeomorphism between $A \subset \mathbb{R}^n$ and $B = F(A)$, then

$$\int_{F(A)} dy = \int_A |J_F(x)|\, dx\,.$$

If injectivity does not hold, then

$$\int_{F(A)} dy \leq \int_A |J_F(x)|\, dx\,. \tag{7}$$

The strict inequality might hold[3]. We apply (7) with $A = \Gamma_u$ and $F = \nabla u$, and we obtain

$$|\nabla u(\Gamma_u)| \leq \int_{\Gamma_u} |J_{\nabla u}(x)|\, dx = \int_{\Gamma_u} |\det(D^2 u(x))|\, dx.$$

If $x \in \Gamma_u$ then, by Lemma 0.2.2, $D^2u(x)$ is a symmetric and positive semidefinite matrix. Thus, its eigenvalues $\lambda_i(x)$ $(i = 1, ..., n)$ are all non-negative and hence so is its determinant, since

$$\det(D^2 u(x)) = \lambda_1(x) \cdot ... \cdot \lambda_n(x).$$

[3]Example: in $\mathbb{R}^2 \approx \mathbb{C}$ consider the map $z \mapsto F(z) = z^2$ and A = unit disc. Then $F(A) = A$, $J_F(z) = 4|z|^2$ and $\int_A J_F(z)\, dz = 2|A|$. In general the *area formula* states that f $F \in C^1(\overline{\Omega}, \mathbb{R}^n)$ (ore more generally if F is Lipschitz continuous) then $\int_{\mathbb{R}^n} \#\{A \cap F^{-1}(y)\}\, dy = \int_A J_F(x)\, dx \quad \forall A \subset \Omega$, A measurable, where $\#(E)$ denotes the counting measure of a set E and $J_F(x) = |\det(DF(x))|$. Notice that $\#\{A \cap F^{-1}(y)\} \neq 0$ if and only if there exists $x \in A$ such that $F(x) = y$ and then $\#\{A \cap F^{-1}(y)\} \geq \chi_{F(A)}(y)$ $\forall y \in \mathbb{R}^n$ where χ_E denotes the characteristic function of a set E.

Moreover, by the classical inequality between the arithmetic and geometric means

$$\lambda_1 \cdot ... \cdot \lambda_n \leq \left(\frac{\lambda_1 + ... + \lambda_n}{n} \right)^n,$$

we also get that

$$\det(D^2 u(x)) \leq \left(\frac{\mathrm{tr}(D^2 u(x))}{n} \right)^n$$

where $\mathrm{tr}(A)$ denotes the trace of a matrix A. But $\mathrm{tr}(D^2 u(x)) = \Delta u(x)$. Then, by (2), $\Delta u(x) = c$ with c given by (3). Hence

$$\omega_n \leq \int_{\Gamma_u} \left(\frac{|\partial\Omega|}{n|\Omega|} \right)^n dx = \left(\frac{|\partial\Omega|}{n|\Omega|} \right)^n |\Gamma_u| \leq \frac{|\partial\Omega|^n}{n^n |\Omega|^{n-1}}$$

that is (1).

0.3 Extremals of the isoperimetric inequality

When $\Omega = B_1$ is the unit ball, then the solution of the Neumann problem (2) is $u(x) = \frac{|x|^2}{2}$ and in the previous proof all the inequalities are in fact equalities. In fact, that proof can be used to show that balls are the only smooth domains for which equality occurs in (1). Let Ω be a bounded smooth domain in $\mathbb{R}^n$ such that equality holds in (1). Roughly speaking, the domain Ω cannot have holes, otherwise, filling them (namely, considering the union of Ω with the bounded components of $\complement\Omega$), the ratio $|\partial\Omega|^n/|\Omega|^{n-1}$ becomes stricly lower and this is impossible. Saying that Ω has no hole means that $\partial\Omega$ is connected. Traking back to the proof, if equality holds in (1), then

$$\begin{aligned} \omega_n \leq |\nabla u(\Gamma_u)| &\leq \int_{\Gamma_u} \det(D^2 u(x))\, dx \leq \int_{\Gamma_u} \left(\frac{\mathrm{tr}(D^2 u(x))}{n} \right)^n dx \\ &\leq \int_{\Omega} \left(\frac{\mathrm{tr}(D^2 u(x))}{n} \right)^n dx = \int_{\Omega} \left(\frac{|\partial\Omega|}{n|\Omega|} \right)^n dx = \frac{|\partial\Omega|^n}{n^n |\Omega|^{n-1}} = \omega_n. \end{aligned}$$

In particular

$$\int_{\Gamma_u} \left(\frac{\mathrm{tr}(D^2 u(x))}{n} \right)^n dx = \int_{\Omega} \left(\frac{\mathrm{tr}(D^2 u(x))}{n} \right)^n dx$$

and since $\mathrm{tr}(D^2 u(x)) = \Delta u(x) = c$ and $\Omega \supseteq \Gamma_u$, it must be $|\Omega \setminus \Gamma_u| = 0$ and so, in particular, Γ_u is dense in Ω. But $u \in C^1(\overline{\Omega})$, hence Γ_u is closed in Ω and then $\Gamma_u = \Omega$. Moreover it must be $\det(D^2 u(x)) = (\mathrm{tr}(D^2 u(x))/n)^n$, i.e., the arithmetic mean and the geometric mean are equal. The occurence of this fact implies that all the eigenvalues are equal. Then $D^2 u(x) = \lambda(x) I$, where I is the identity matrix. In fact $\lambda(x) = \mathrm{tr}(D^2 u(x))/n = |\partial\Omega|/(n|\Omega|) =: \lambda$. Thus

$$D^2 u(x) = \lambda I \ \forall x \in \Omega. \tag{8}$$

Finally, $|\nabla u(\Gamma_u)| = \omega_n$ implies that $B_1(0)$ is dense in $\nabla u(\Gamma_u)$. Hence $|\nabla u(x)| \leq 1$ for all $x \in \overline{\Omega}$. At every point $x \in \partial\Omega$ one has

$$1 = \frac{\partial u}{\partial \nu}(x) = \nabla u(x) \cdot \nu(x).$$

Therefore for any tangent vector τ at x one has $\nabla u(x) \cdot \tau = 0$, namely u has to be constant on $\partial\Omega$ (which is connected, as observed before). Changing u by a constant if necessary we may assume that u solves the over-determined problem

$$\begin{cases} \Delta u = c & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega. \end{cases}$$

Let $x_0 \in \overline{\Omega}$ be the minimum point of u. Since $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$, it must be $x_0 \in \Omega$. Then $R := \text{dist}(x_0, \partial\Omega) > 0$ and $B_R(x_0) \subset \Omega$ and there exists $x_1 \in \partial\Omega \cap \partial B_R(x_0)$. By Taylor's theorem, for every $x \in \overline{B_R(x_0)}$ there exists a point ξ in the segment $[x_0, x]$ such that

$$u(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + \frac{1}{2}\langle D^2 u(\xi)(x - x_0), x - x_0\rangle$$

and then

$$u(x) = u(x_0) + \frac{\lambda}{2}|x - x_0|^2 \quad \forall x \in \overline{B_R(x_0)}$$

because $\nabla u(x_0) = 0$ (x_0 is a minimum point) and by (8). Since u in $\overline{B_R(x_0)}$ depends just on the distance $|x - x_0|$ and vanishes at x_1 we obtain that

$$u = 0 \text{ on } \partial B_R(x_0).$$

Finally, we apply the maximum principle: if $\Delta u > 0$ in Ω and $u|_{\partial\Omega} = 0$ then either $u \equiv 0$ in Ω or $u < 0$ in Ω. The first case cannot occur because $\frac{\partial u}{\partial \nu} \neq 0$ on $\partial\Omega$. Hence $u < 0$ in Ω. Thus it must be $\partial B_R(x_0) = \partial\Omega$ and then $\Omega = B_R(x_0)$. □

Remark 0.9 (Isoperimetric deficit and Fraenkel asymmetry)**.** *For every bounded domain Ω in $\mathbb{R}^n$ (regular enough), one introduces the* isoperimetric deficit *of Ω given by*

$$D(\Omega) = \frac{|\partial\Omega|}{n\omega_n^{1/n}|\Omega|^{(n-1)/n}} - 1$$

and the symmetry deficit *or* Fraenkel asymmetry *of Ω, defined by*

$$\lambda(\Omega) = \min_{x \in \mathbb{R}^n} \frac{|\Omega \div B_r(x)|}{r^n} \quad \textit{with } r \textit{ such that } |B_r| = |\Omega|.$$

In 2008 Fusco, Maggi and Pratelli proved that there exists a constant $C(n)$ independent of Ω such that

$$\lambda(\Omega) \leq C(n)\sqrt{D(\Omega)}.$$

Appendix: proof of the Poincaré inequality

Proof. Arguing by contradiction, assume that there exists a sequence $(u_k) \subset W^{1,p}(\Omega)$ such that

$$\int_\Omega |u_k - [u_k]|^p > k \int_\Omega |\nabla u_k|^p \quad \forall k \in \mathbb{N}.$$

Set

$$v_k = \frac{u_k - [u_k]}{\|u_k - [u_k]\|_{L^p}}.$$

It holds that

$$[v_k] = 0, \quad \|v_k\|_{L^p} = 1, \quad \|\nabla v_k\|_{L^p} < \frac{1}{k}.$$

Thus the sequence (v_k) is bounded in $W^{1,p}(\Omega)$, which is reflexive for $p \in (1, \infty)$. Hence it admits a subsequence, still denoted (v_k), weakly converging to some $v \in W^{1,p}(\Omega)$. By the Rellich theorem, $v_k \to v$ strongly in $L^p(\Omega)$ and then also $L^1(\Omega)$. Therefore $[v] = 0$, $\|v\|_{L^p} = 1$ and for every $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega v \frac{\partial \varphi}{\partial x_i} = \lim_{k\to\infty} \int_\Omega v_k \frac{\partial \varphi}{\partial x_i} = - \lim_{k\to\infty} \int_\Omega \varphi \frac{\partial v_k}{\partial x_i} = 0\,.$$

By definition and uniqueness of the weak derivative, one has that $\frac{\partial v}{\partial x_i} = 0$ a.e., for all $i = 1, ..., n$. Then, since Ω is connected, v is constant and, since $[v] = 0$, it must be $v = 0$, contrary to $\|v\|_{L^p} = 1$. □