14 The Dirichlet problem for the Lane-Emden equation

We are interested in the following Dirichlet problem

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{14.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $p > 1$. The equation in (14.1) is called Lane-Emden equation, after astrophysicists J. Lane and R. Emden, and constitutes a model in the study of stellar structure. Problem (14.1) is also introduced in some studies of flame propagation (Gelfand, Some problems in the theory of quasilinear equations, 1963).

The Lane-Emden equation is semilinear because the nonlinearity is just on u. The nonlinearity is superlinear, because $p > 1$.

14.1 Existence results in the subcritical case

Problem (14.1) is variational, in the sense that its solutions can be found as critical points of a suitable functional in a certain Banach space. More precisely, consider the Hilbert space $W_0^{1,2}(\Omega)$, endowed with its inner product

$$\langle u, v \rangle_{W_0^{1,2}} = \int_\Omega \nabla u \cdot \nabla v \, dx$$

and define the *energy functional*

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} \, dx \quad \forall u \in W_0^{1,2}(\Omega).$$

The energy functional is well defined and Fréchet-differentiable at every $u \in W_0^{1,2}(\Omega)$ and

$$E'(u)[h] = \int_\Omega \nabla u \cdot \nabla h \, dx - \int_\Omega |u|^{p-1} u h \, dx \quad \forall h \in W_0^{1,2}(\Omega).$$

This is due to the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ which holds true for $p \in \left[1, \frac{N+2}{N-2}\right]$ in dimension $N \geq 3$, or for all $p \geq 1$ in dimension $N = 1, 2$. Notice that for $N \geq 3$, the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact if $p+1 < 2^*$ where $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent. This corresponds to take $p < \frac{N+2}{N-2}$. Therefore the nonlinearity is called *subcritical*. Observe that

$$E'(u) = 0 \;\Leftrightarrow\; u \text{ weak solution to } \begin{cases} \Delta u + |u|^{p-1} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence we are interested in finding positive critical points of E.

Remark 14.1. *Looking for critical points of E as global minimizers of E is not possible because $\inf_{u\in W_0^{1,2}(\Omega)} E(u) = -\infty$. Indeed, fixing $u \in W_0^{1,2}(\Omega) \setminus \{0\}$, one has that*

$$E(tu) = \frac{t^2}{2}\int_\Omega |\nabla u|^2\,dx - \frac{t^{p+1}}{p+1}\int_\Omega |u|^{p+1}\,dx \to -\infty \quad \text{as } t \to \infty,$$

because $p > 1$. Also a maximization approach fails, since $\sup_{u\in W_0^{1,2}(\Omega)} E(u) = \infty$. To show this, we take the increasing sequence (λ_n) of eigenvalues of the Dirichlet-Laplacian operator on Ω, a sequence $(\varphi_n) \subset W_0^{1,2}(\Omega)$ of corresponding eigenfunctions, and for every n we choose $t_n > 0$ such that

$$E(t_n\varphi_n) = \max_{t>0} E(t\varphi_n).$$

We claim that $E(t_n\varphi_n) \to \infty$. Indeed, considering the case of dimension $N \geq 3$, since $2 < p+1 < 2^*$, we can write

$$p + 1 = 2\theta + (1-\theta)2^* \quad \text{for some } \theta \in (0,1).$$

Thus, by Hölder's inequality,

$$\int_\Omega |\varphi_n|^{p+1}\,dx \leq \left(\int_\Omega \varphi_n^2\,dx\right)^\theta \left(\int_\Omega |\varphi_n|^{2^*}\,dx\right)^{1-\theta}. \tag{14.2}$$

Moreover, since φ_n is an eigenfunction,

$$\int_\Omega \varphi_n^2\,dx = \lambda_n^{-1}\int_\Omega |\nabla\varphi_n|^2\,dx$$

and, by the Sobolev embedding, one has

$$\int_\Omega |\varphi_n|^{2^*}\,dx \leq C\left(\int_\Omega |\nabla\varphi_n|^2\,dx\right)^{2^*/2}$$

for some constant C independent of n. Hence, from (14.2) it follows that

$$\int_\Omega |\varphi_n|^{p+1}\,dx \leq C^{1-\theta}\lambda_n^{-\theta}\left(\int_\Omega |\nabla\varphi_n|^2\,dx\right)^{\frac{p+1}{2}}.$$

As a consequence,

$$E(t\varphi_n) \geq \frac{t^2}{2}\int_\Omega |\nabla\varphi_n|^2\,dx - C^{1-\theta}\lambda_n^{-\theta}\frac{t^{p+1}}{p+1}\left(\int_\Omega |\nabla\varphi_n|^2\,dx\right)^{\frac{p+1}{2}}$$

and then

$$\max_{t>0} E(t\varphi_n) \geq \max_{t>0} \left(\frac{t^2}{2} - C^{1-\theta} \lambda_n^{-\theta} \frac{t^{p+1}}{p+1} \right) = \left(\frac{1}{2} - \frac{1}{p+1} \right) C^{\frac{2}{2-2^*}} \lambda_n^{\frac{2\theta}{p-1}} \to \infty$$

since $p > 1$, $\theta > 0$ *and* $\lambda_n \to \infty$ *as* $n \to \infty$. *In dimension* $N = 1, 2$ *one can repeat the argument taking any* $q > p$ *instead of* 2^*.

In order to find a solution to (14.1) with a variational approach, we can take advantage of the fact that the nonlinearity is homogeneous, and we study the following *constrained* minimization problem:

$$\inf_{u \in M} \|\nabla u\|_{L^2}^2 \quad \text{where } M = \{u \in W_0^{1,2}(\Omega) \mid \|u\|_{L^{p+1}} = 1\}.$$

The strategy can be outlined by these steps:

(1) $\exists v \in M$ such that $v \geq 0$ and $\|\nabla v\|_{L^2}^2 = \inf_{u \in M} \|\nabla u\|_{L^2}^2$,

(2) $\exists \alpha > 0$ such that αv is a weak solution to (14.1),

(3) every weak solution of (14.1) is a classical solution (if Ω is regular enough),

(4) if u is a classical solution to

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega \\ u \geq 0, \ u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{14.3}$$

then $u > 0$ in Ω.

In this way we arrive to show:

Theorem 14.2. *If* Ω *is a bounded domain in* $\mathbb{R}^N$ *of class* C^2 *and* $1 < p < \frac{N+2}{N-2}$ *for* $N \geq 3$, *or* $p > 1$ *for* $N = 1, 2$, *then problem (14.1) admits a classical solution.*

Proof of (1) Set $\mu = \inf_{u \in M} \|\nabla u\|_{L^2}^2$. Take a sequence $(u_n) \subset M$ such that $\|\nabla u_n\|_{L^2}^2 \to \mu$. Since $\|\nabla u\|_{L^2}$ is a norm in $W_0^{1,2}(\Omega)$, the sequence (u_n) is bounded in $W_0^{1,2}(\Omega)$. Hence it admits a subsequence, still denoted (u_n), which converges weakly in $W_0^{1,2}(\Omega)$ to some v. Since $p + 1 < 2^*$, by Rellich's Theorem, $u_n \to v$ strongly in $L^{p+1}(\Omega)$. Thanks to the estimate $\big| \|u_n\|_{L^{p+1}} - \|v\|_{L^{p+1}} \big| \leq \|u_n - v\|_{L^{p+1}}$, we deduce that $v \in M$ and then $\|\nabla v\|_{L^2}^2 \geq \mu$. Moreover from the weak lower semicontinuity of the norm $\|u\|_{W_0^{1,2}} = \|\nabla u\|_{L^2}$, it follows that $\|\nabla v\|_{L^2}^2 \leq \liminf \|\nabla u_n\|_{L^2}^2 = \mu$. Therefore $\|\nabla v\|_{L^2}^2 = \mu$, namely v is a minimum point. Since $\big|\nabla |v|\big| =$

$|\nabla v|$ a.e. in Ω, also $|v|$ is a minimum point and it is a non negative function. Thus Step (1) is proved.

Proof of (2) Let

$$F(u) = \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{p+1}}^2} \quad \forall u \in W_0^{1,2}(\Omega) \setminus \{0\}$$

and observe that $W_0^{1,2}(\Omega)\setminus\{0\}$ is open in $W_0^{1,2}(\Omega)$ and F is differentiable at every $u \in W_0^{1,2}(\Omega)\setminus\{0\}$, with

$$F'(u)[h] = \frac{2\|u\|_{L^{p+1}}^2 \int_\Omega \nabla u \cdot \nabla h\, dx - 2\|\nabla u\|_{L^2}^2 \int_\Omega |u|^{p-1} u h\, dx}{\|u\|_{L^{p+1}}^2} \quad \forall h \in W_0^{1,2}(\Omega).$$

For every $u \in W_0^{1,2}(\Omega)$, $u \neq 0$ one has that $F(u) = \|\nabla \hat{u}\|_{L^2}^2 \geq \mu$ where $\hat{u} = u/\|u\|_{L^{p+1}}^2$. Hence if $v \in M$ is a minimizer for $\|\nabla u\|_{L^2}^2$ in M, it is also a minimizer for F in $W_0^{1,2}(\Omega) \setminus \{0\}$. By Fermat's Theorem, $F'(v) = 0$, namely

$$\|v\|_{L^{p+1}}^2 \int_\Omega \nabla v \cdot \nabla h\, dx = \|\nabla v\|_{L^2}^2 \int_\Omega |v|^{p-1} v h\, dx \quad \forall h \in W_0^{1,2}(\Omega).$$

Since $v \in M$, $v \geq 0$, and $\|\nabla v\|_{L^2}^2 = \mu$, it remains

$$\int_\Omega \nabla v \cdot \nabla h\, dx = \mu \int_\Omega v^p h\, dx \quad \forall h \in W_0^{1,2}(\Omega).$$

Then the mapping $w = \alpha v$, with $\alpha = \mu^{\frac{1}{p-1}}$, solves

$$\int_\Omega \nabla w \cdot \nabla h\, dx = \int_\Omega w^p h\, dx \quad \forall h \in W_0^{1,2}(\Omega)$$

namely it is a weak solution to (14.1).

Proof of (3) Regularity of weak solutions is based on these fundamental tools: Sobolev embeddings, L^p elliptic regularity theory and Schauder regularity theory. Let us recall them briefly:

- **Sobolev embeddings**

  Let Ω be a bounded, Lipschitz domain in $\mathbb{R}^N$. Let $k \in \mathbb{N}$ and $p \geq 1$.

  - If $kp < N$ then $W^{k,p}(\Omega) \subset L^q(\Omega)$ for all $q \in \left[1, \frac{Np}{N-kp}\right]$.
  - If $kp = N$ then $W^{k,p}(\Omega) \subset L^q(\Omega)$ for all $q \in [1, \infty)$.

– If $kp > N$ and $k - \frac{N}{p} = m + \alpha$ with $m \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0,1)$, then $W^{k,p}(\Omega) \subset C^{m,\alpha}(\overline{\Omega})$, with continuous inclusion, in the sense that for every $u \in W^{k,p}(\Omega)$ there exists $v \in C^{m,\alpha}(\overline{\Omega})$ such that $u = v$ a.e. in Ω.

In all the cases, the inclusions are continuous.

- L^p **elliptic regularity theory**
  - *Local regularity:* if $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution of $-\Delta u = f$ and $f \in L^p_{loc}(\Omega)$ for some $p \in (1,\infty)$, then $u \in W^{2,p}_{loc}(\Omega)$.
  - *Global regularity:* Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2 (*). If $u \in W^{1,2}_0(\Omega)$ is a weak solution of $-\Delta u = f$ and $f \in L^p(\Omega)$ for some $p \in (1,\infty)$, then $u \in W^{2,p}(\Omega)$.
- **Schauder regularity theory**
  - *Interior regularity:* If $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution of $-\Delta u = f$ and $f \in C^{0,\alpha}_{loc}(\Omega)$ for some $0 < \alpha < 1$, then $u \in C^{2,\alpha}_{loc}(\Omega)$.
  - *Global regularity:* Let Ω be a bounded domain in $\mathbb{R}^N$, of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$. If $u \in W^{1,2}_0(\Omega)$ is a weak solution of $-\Delta u = f$ and $f \in C^{0,\alpha}(\overline{\Omega})$ then $u \in C^{2,\alpha}(\overline{\Omega})$.

Let now $u \in W^{1,2}_0(\Omega)$ be a weak solution of $\Delta u + u^p = 0$ in Ω and assume Ω of class C^2 (*). It is enough to show that

$$u \in L^q(\Omega) \quad \forall q \in (1,\infty). \tag{14.4}$$

Indeed:

Lemma 14.3. *If $u \in W^{1,2}_0(\Omega)$ is a weak solution of $\Delta u + |u|^{p-1}u = 0$ in Ω satisfying (14.4), then u is a classical solution of the Dirichlet problem*

$$\begin{cases} \Delta u + |u|^{p-1}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Set $f = |u|^{p-1}u$. Since u satisfies (14.4), also $f \in L^q(\Omega)$ for all $q \in (1,\infty)$ and then, by the L^p elliptic regularity theory, $u \in W^{2,q}(\Omega)$ for all $q \in (1,\infty)$. Fixing $\alpha \in (0,1)$, let $q > 1$ be such that $1 + \alpha = 2 - \frac{N}{q}$. By the Sobolev embeddings (with $k = 2$ and $m = 1$), $u \in C^{1,\alpha}(\overline{\Omega})$. In particular $f \in C^{0,\alpha}_{loc}(\Omega)$, and then, by the Schauder regularity theory, $u \in C^{2,\alpha}_{loc}(\Omega)$. In particular $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Since Ω is of class C^1 and $u \in W^{1,2}_0(\Omega)$ one has that $u(x) = 0$ for every $x \in \partial\Omega$ (use Theorem 9.17 of Brezis: Functional Analysis, Sobolev Spaces and Partial

(*) Of class $C^{1,1}$ would be sufficient.

Differential Equations, Springer 2011). Moreover one can apply the divergence theorem and deduce that

$$0 = \int_\Omega (-\nabla u \cdot \nabla h + fh)\, dx = \int_\Omega (\Delta u + f) h \, dx \quad \forall h \in C_c^\infty(\Omega).$$

Then, using a standard result, one concludes that $\Delta u + f = 0$ a.e. in Ω and finally, since $\Delta u + f$ is continuous, $\Delta u(x) + f(x) = 0$ for all $x \in \Omega$. $\square$

Now let us show that u satisfies (14.4). This is immediately true in dimensions $N = 1, 2$ because of the Sobolev embeddings: for $N = 1, 2$ one has that $W^{1,2}(\Omega) \subset L^q(\Omega)$ for all $q \in [1, \infty)$. Consider the case of dimension $N \geq 3$. By the Sobolev embedding, $W^{1,2}(\Omega) \subset L^{2^*}(\Omega)$, where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. Hence $u \in L^{2^*}$ and consequently $f = u^p \in L^{2^*/p}$. Set $p_1 = \frac{2^*}{p}$ and observe that $p_1 > 1$ because $p < \frac{N+2}{N-2} < 2^*$ in dimension $N \geq 3$. Then use the L^p elliptic regularity theory: since $f \in L^{p_1}$ with $p_1 > 1$, then $u \in W^{2,p_1}$. If $2p_1 \geq N$, then, by the Sobolev embeddings, $u \in L^q$ for all $q \in (1, \infty)$ and we are done. Otherwise $2p_1 < N$ and, again by the Sobolev embeddings, $u \in L^{q_1}$, where

$$q_1 = \frac{Np_1}{N - 2p_1}.$$

Observe that $q_1 > 2^*$. Indeed, since $p < 2^* - 1$,

$$q_1 = \frac{N \cdot 2^*}{Np - 2 \cdot 2^*} > \frac{N \cdot 2^*}{N \cdot 2^* - N - 2 \cdot 2^*} = 2^*$$

(use the definition of $2^* = \frac{2N}{N-2}$). As $u \in L^{q_1}$, then $f = u^p \in L^{q_1/p}$. Setting $p_2 = \frac{q_1}{p}$, one has that $p_2 > \frac{2^*}{p} > 1$. Hence, by the L^p elliptic regularity theory, $u \in W^{2,p_2}$. If $2p_2 \geq N$, then, by the Sobolev embeddings, u satisfies (14.4) and we are done. Otherwise $2p_2 < N$ and, again by the Sobolev embeddings, $u \in L^{q_2}$, where

$$q_2 = \frac{Np_2}{N - 2p_2}.$$

We can repeat this "boot-strap" argument: starting with $q_0 = 2^*$, we define exponents q_j and p_j such that

$$q_j = \frac{Np_j}{N - 2p_j} \quad \text{and} \quad p_{j+1} = \frac{q_j}{p}. \tag{14.5}$$

If $2p_j \geq N$ for some j, then (14.4) holds. Otherwise $2p_j < N$ for every $j \in \mathbb{N}$. We show that this cannot occur, arguing by contradiction. We claim that $q_j > 2^*$ for all j. This has been proved for $j = 1$. Assuming that it is true for a fixed j, using (14.5), we have that

$$q_{j+1} = \frac{Nq_j}{Np - 2q_j} > \frac{Nq_j}{N \cdot 2^* - N - 2 \cdot 2^*} = q_j > 2^*. \tag{14.6}$$

We also obtained that (q_j), and then also (p_j) is an increasing sequence. But we also know that $2p_j < N$ for every $j \in \mathbb{N}$. Therefore (p_j), and then also (q_j) are convergent sequences. But from (14.6) it follows that

$$\frac{1}{q_j} - \frac{1}{q_{j+1}} = \frac{p-1}{q_j} - \frac{2}{N} < \frac{p-1}{2^*} - \frac{2}{N} = \frac{1}{2^*}\left(p - \frac{N+2}{N-2}\right) < 0$$

which yields a contradiction.

Proof of (4) If u is a classical solution to (14.3) then $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u \geq 0 & \text{in } \Omega \\ u \geq 0,\ u \not\equiv 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The maximum principle immediately implies that $u(x) > 0$ for every $x \in \Omega$. □

Remark 14.4. *(1) With the same procedure one can prove a similar result for existence of solutions to problem*

$$\begin{cases} Lu = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $L = -\Delta + q(x)$, being $q \in C^{0,\alpha}(\overline{\Omega})$) or even when L is any uniformly elliptic operator in the divergence form, with regular coefficients. On Ω and p one assume the same hypotheses as in Theorem 14.2. The proof differs just in the first step, as one considers a minimization problem of the form $\inf_{u\in M} F(u)$, where $F(u)$ is a quadratic form associated to the operator L. If $Lu = -div(A(x)\nabla u) + q(x)u$ then $F(u) = \int_\Omega (\nabla u \cdot (A(u)\nabla u) + q(x)u^2)\,dx$.

(2) The variational approach consisting in minimizing the norm $\|\nabla u\|_{L^2}^2$ in the set $M = \{u \in W_0^{1,2}(\Omega) \mid \|u\|_{L^{p+1}} = 1\}$ is strongly related to the homogeneity character of the nonlinearity. Moreover notice that the constraint M is weakly closed in $W_0^{1,2}(\Omega)$ because p is subcritical. This bound is not just a technical restriction. For p critical or supercritical the situation is rather different and in some cases, for example when Ω is a ball, problem (14.1) has no non zero solution.

14.2 Non existence results in the critical or supercritical case

Here we consider the case of critical or super-critical exponent and we prove a non existence result which, in fact, needs a geometric condition on the domain, expressed according to the following definition.

A domain Ω is called (strictly) star-shaped if there exists a point $x_0 \in \Omega$ such that $(x - x_0) \cdot \nu(x) > 0$ at every $x \in \partial\Omega$.

Theorem 14.5. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. In dimension $N \geq 3$, if Ω is star-shaped and $p \geq \frac{N+2}{N-2}$ then problem (14.1) has no solution $u \in C^2(\overline{\Omega})$.*

The proof of Theorem 14.5 is based on a remarkable identity, due to S. Pohozaev:

Theorem 14.6. *Let Ω be a bounded, regular domain in $\mathbb{R}^N$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then

$$\left(\frac{N}{2} - 1\right) \int_\Omega |\nabla u|^2 \, dx - N \int_\Omega F(u) \, dx + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu) \, d\sigma = 0 \,. \tag{14.7}$$

where $F(t) = \int_0^t f(s) \, ds$ and, for every $x \in \partial\Omega$, $\nu = \nu(x)$ is the outer normal vector to $\partial\Omega$ at x.

The domain Ω is assumed to be regular enough in order that one can draw the outer normal $\nu(x)$ at every $x \in \partial\Omega$.

Dimostrazione. We have that

$$0 = \int_\Omega (\Delta u + f(u))(x \cdot \nabla u) \, dx \,.$$

Moreover $\nabla(F \circ u) = f(u)\nabla u$ and then, since $\operatorname{div}(x\,(F \circ u)) = N\,(F \circ u) + x \cdot \nabla(F \circ u)$, by the divergence theorem,

$$\int_\Omega f(u)(x \cdot \nabla u) \, dx = \int_\Omega x \cdot \nabla(F \circ u) \, dx = -N \int_\Omega F(u) \, dx + \int_{\partial\Omega} (x \cdot \nu) F(u) \, d\sigma \,.$$

Setting $G = (x \cdot \nabla u)\nabla u$, we have that

$$\operatorname{div} G = \nabla(x \cdot \nabla u) \cdot \nabla u + (x \cdot \nabla u)\Delta u$$

and we compute

$$\begin{aligned} \nabla(x \cdot \nabla u) \cdot \nabla u &= u_{x_i} \partial_{x_i} (x_j u_{x_j}) = u_{x_i} (\delta_{ij} u_{x_j} + x_j u_{x_i x_j}) \\ &= |\nabla u|^2 + x_j \partial_{x_j} \left(\frac{|\nabla u|^2}{2} \right) = |\nabla u|^2 + x \cdot \nabla \left(\frac{|\nabla u|^2}{2} \right) \end{aligned}$$

and

$$x \cdot \nabla \left(|\nabla u|^2 \right) = \operatorname{div} \left(x |\nabla u|^2 \right) - N |\nabla u|^2 \, .$$

Then

$$\begin{aligned} \int_\Omega (\Delta u)(x \cdot \nabla u) \, dx &= \int_\Omega \nabla (x \cdot \nabla u) \cdot \nabla u \, dx - \int_\Omega \operatorname{div} G \, dx \\ &= \left(1 - \frac{N}{2} \right) \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu) \, d\sigma - \int_{\partial\Omega} G \cdot \nu \, d\sigma \, . \end{aligned}$$

Hence

$$\begin{aligned} \left(\frac{N}{2} - 1 \right) \int_\Omega |\nabla u|^2 \, dx - N \int_\Omega F(u) \, dx = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu) \, d\sigma \\ - \int_{\partial\Omega} (x \cdot \nabla u)(\nu \cdot \nabla u) \, d\sigma - \int_{\partial\Omega} (x \cdot \nu) F(u) \, d\sigma \, . \end{aligned}$$

Since $u = 0$ on $\partial\Omega$ and $F(0) = 0$, the last integral vanishes. Moreover since the gradient is orthogonal to level sets and u is constant on $\partial\Omega$, then $\nabla u = \pm |\nabla u| \nu$ on $\partial\Omega$. In both cases, $(x \cdot \nabla u)(\nu \cdot \nabla u) = |\nabla u|^2 (x \cdot \nu)$. Therefore (14.7) follows. □

Proof of Theorem 14.5. Since the equation does not contain an explicit dependence on x, we may assume that Ω is star-shaped with respect to the origin, so that $x \cdot \nu > 0$ on $\partial\Omega$. By the divergence theorem and since $u = 0$ on $\partial\Omega$, we have that

$$\int_\Omega |\nabla u|^2 \, dx = - \int_\Omega u \Delta u \, dx = \int_\Omega u^{p+1} \, dx \, . \tag{14.8}$$

We apply Theorem 14.6 with $f(t) = t^p$ and then $F(t) = \frac{t^{p+1}}{p+1}$. Hence (14.7) becomes

$$\left(\frac{N}{2} - 1 \right) \int_\Omega |\nabla u|^2 \, dx - N \int_\Omega \frac{u^{p+1}}{p+1} \, dx = -\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu) \, d\sigma \, .$$

and then, using (14.8),

$$\left(\frac{N}{2} - 1 - \frac{N}{p+1} \right) \int_\Omega u^{p+1} \, dx = -\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu) \, d\sigma \, .$$

We can apply the Hopf boundary point Lemma. Since $-\Delta u > 0$ in Ω, $u = 0$ on $\partial\Omega$ and $u \not\equiv 0$, we have that $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$ and then

$$\int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu) \, d\sigma > 0 \, .$$

Therefore, as $u > 0$ in Ω, it must be $\frac{N}{2} - 1 - \frac{N}{p+1} < 0$, namely $p < \frac{N+2}{N-2}$. □