15 Semilinear elliptic problems – The compact case

We are interested in the following Dirichlet problem

$$\begin{cases} -\Delta u + q(x)u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $q \in L^\infty(\Omega)$, $q \geq 0$ and $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function whose dependence on u is superlinear and subcritical. More precisely, we assume:

(g_1) $\exists p \in (1, \frac{N+2}{N-2})$ and $\exists c_1, c_2 > 0$ such that $c_1|u|^p \leq |g(x,u)| \leq c_2|u|^p \ \forall u \in \mathbb{R}$, for a.e. $x \in \Omega$;

(g_2) $\exists \beta > 2$ such that $0 < \beta G(x,u) \leq g(x,u)u \ \forall u \in \mathbb{R} \setminus \{0\}$, for a.e. $x \in \Omega$, where $G(x,u) = \int_0^u g(x,s)\,ds$.

Remark 15.1 *1. By (g_1) one has that $g(x,0) = 0 \ \forall x$. Hence $u \equiv 0$ solves (1). Indeed we look for nontrivial (i.e. non null) solutions. In particular, we are interested just in weak solutions. To this aim $g \in L^\infty$ is enough. In order to pass from weak to classical solutions by means of regularity theory, one assumes that q and G are Hölder-continuous.*

2. Assumptions (g_1)–(g_2) are fulfilled by $g(x,u) = b(x)|u|^{p-1}u$, with $b \in C(\Omega)$ such that $b_1 \leq b(x) \leq b_2 \ \forall x \in \Omega$, for some $b_1, b_2 > 0$. In particular, if $g(x,u) = b(x)|u|^{p-1}u$, then $G(x,u) = \frac{1}{p+1}b(x)|u|^{p+1}$ and (g_2) is satisfied with $\beta = p+1$ (note that $\beta > 2$, since $p > 1$).

3. The assumption (g_1) is a "pincing" condition on g and says that g is superlinear with respect to u, because $p > 1$. Note that the linear behaviour is given by the term $q(x)u$ at the left hand side. The upper bound $p < \frac{N+2}{N-2}$ is asked just in dimension $N \geq 3$; for $N = 1, 2$ no upper bound on p is needed). Since $\frac{N+2}{N-2} = 2^* - 1$ where $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent, in view of (g_1) we are dealing with a nonlinearity with a subcritical growth.

4. The assumption (g_2) is called "superquadratic" condition and plays a technical role in the argument. In particular it implies that $G(x,u) \leq C_1|u|^\beta$ for $|u| \leq 1$, whereas $G(x,u) \geq C_2|u|^\beta$ for $|u| \geq 1$. To check it, observe that $\frac{d}{dt}[t^{-\beta}G(x,tu)] \geq 0$ for $t > 0$, because of (g_2).

We study problem (1) using variational methods. In particular, we look for weak solutions of (1) as critical points of a suitable functional on the Sobolev space $W_0^{1,2}(\Omega)$. More precisely, let us introduce the functional $E\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ defined as follows:

$$E(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + q(x)u^2)\,dx - \int_\Omega G(x,u)\,dx\,. \tag{2}$$

Since $q \in L^\infty(\Omega)$ and $g \in C(\Omega \times \mathbb{R})$ satisfies (g_1), we know that E is well defined and of class C^1 in the space $W_0^{1,2}(\Omega)$. Moreover

$$E'(u)[h] = \int_\Omega (\nabla u \cdot \nabla h + q(x)uh)\,dx - \int_\Omega g(x,u)h\,dx \quad \forall u, h \in W_0^{1,2}(\Omega).$$

In particular critical points of E correspond to weak solutions to problem (1). We will show that

Theorem 15.2 *Under the above assumptions on Ω and q and if $g \in C(\Omega \times \mathbb{R})$ satisfies (g_1)–(g_2), then problem (1) admits a nontrivial weak solution.*

In fact, existence of nontrivial weak solutions (1) holds true under more general assumptions than (g_1)–(g_2) (see, for instance, the original paper by A. Ambrosetti and P.H. Rabinowitz, J. Funct. Anal. **14** (1973)). We limit ourselves to this case because it contains all the main ideas avoiding too much technicalities.

As we alredy saw in the purely homogeneous case $g(x,u) = |u|^{p-1}u$, the functional E is not bounded both from below and from above. No hope to find critical points by studying global minimization (or maximization) problems. We point out that we can endow the space $W_0^{1,2}(\Omega)$ with the equivalent norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + q(x)u^2)\,dx$$

Indeed, since $q \in L^\infty$ and $q \geq 0$, we have

$$\begin{aligned}\|u\|^2_{W_0^{1,2}} = \int_\Omega |\nabla u|^2\,dx &\leq \int_\Omega (|\nabla u|^2 + q(x)u^2)\,dx \\ &\leq \max\{1, \|q\|_{L^\infty}\} \int_\Omega (|\nabla u|^2 + u^2)\,dx \leq C\|u\|^2_{W_0^{1,2}}\end{aligned}$$

by the Poincaré inequality. Moreover, we observe that the origin is a (trivial) critical point of E. In fact:

Lemma 15.3 *The origin is a strict local minimum of E. In particular $\exists r > 0$ such that $E(u) \geq \alpha > 0$ for every $u \in W_0^{1,2}(\Omega)$ with $\|u\| = r$. Moreover there exists $u_1 \in W_0^{1,2}(\Omega)$ with $\|u_1\| > r$ and $E(u_1) \leq 0$.*

Proof. Indeed, from (g_1) and since $g(x,u)u > 0$, one has

$$\frac{c_1}{p+1}|u|^{p+1} \leq G(x,u) \leq \frac{c_2}{p+1}|u|^{p+1} \quad \forall u \in \mathbb{R} \quad \text{and a.e. } x \in \Omega. \tag{3}$$

Since $p+1 \leq 2^*$, the space $W_0^{1,2}(\Omega)$ is embedded into L^{p+1}, that is, there exists $C > 0$ such that $\|u\|_{L^{p+1}} \leq C\|u\|$ $\forall u \in W_0^{1,2}$. Then

$$\begin{aligned}E(u) = \frac{1}{2}\|u\|^2 - \int_\Omega G(x,u)\,dx &\geq \frac{1}{2}\|u\|^2 - \frac{c_2}{p+1}\|u\|^{p+1}_{L^{p+1}} \\ &\geq \frac{1}{2}\|u\|^2 - C\|u\|^{p+1} = \frac{1}{2}\|u\|^2\left(1 - C\|u\|^{p-1}\right) \geq \frac{r^2}{4}\end{aligned}$$

for $\|u\| = r$ and $r > 0$ small enough. This shows the first part. In order to prove the second statement, we use the lower bound in (3) getting that

$$E(tu) \leq \frac{t}{2}\|u\|^2 - c_1 \frac{t^{p+1}}{p+1}\|u\|^{p+1}_{L^{p+1}}.$$

Fixing $u \in W^{1,2}_0$, $u \neq 0$ and taking $t > 0$ large enough, the function tu satisfies the required properties. $\square$

Lemma 15.3 states that the functional E has a so-called mountain pass geometry: there are two points in the space, which are the origin and u_1 where the functional takes non positive value and following any path joining them, the functional has to overcome the level $\alpha > 0$. This suggests that a critical point of saddle type might be found at the mountain pass level. In fact, this idea has to be precised and developed, as follows.

Definition 15.4 *Let X be a Banach space and let $E\colon X \to \mathbb{R}$ be a differentiable functional. A number $c \in \mathbb{R}$ is*

- *a critical level of E if there exists $u \in X$ such that $E'(u) = 0$ and $E(u) = c$;*
- *an asymptotic critical level of E if there exists a sequence $(u_n) \subset X$ such that $\|E'(u_n)\| \to 0$ and $E(u_n) \to c$. Such a sequence is called Palais-Smale sequence at level c, shortly $(PS)_c$-sequence.*

We say that the functional E satisfies the Palais-Smale condition at level c (shortly, $(PS)_c$ condition) if every $(PS)_c$-sequence admits a strongly convergent subsequence.

Theorem 15.5 (Mountain Pass Theorem) *Let X be a Banach space and let $E\colon X \to \mathbb{R}$ be a functional of class C^1 satisfying:*

(i) $\exists r > 0$ *such that* $E(u) \geq \alpha > E(0)$ *for all* $u \in X$ *with* $\|u\| = r$,

(ii) $\exists u_1 \in X$ *with* $\|u_1\| > r$ *and* $E(u_1) \leq E(0)$.

Then, setting

$$\Gamma = \{\gamma \in C([0,1]; X) \mid \gamma(0) = 0,\ \gamma(1) = u_1\} \quad \text{and} \quad c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)), \tag{4}$$

one has that $c \geq \alpha$ and c is an asymptotic critical level of E. If, in addition, E satisfies $(PS)_c$ condition, then c is a critical level of E.

The set Γ in (4) is the class of the mountain pass paths, whereas the value c is the mountain pass level. Notice that the mountain pass geometry in general is not sufficient to guarantee the existence of a critical point at the mountain pass level. The additional condition $(PS)_c$ is a compactness property ensuring that the mountain pass level is a critical level.

The Mountain Pass Theorem is the prototype of minimax theorems, concerning the existence of saddle-type critical points. Among the several texts dealing with this topic, we mention:

– M. Stuwe: Variational Methods, Springer, 1990;

– P.H. Rabinowitz: Minimax methods in critical point theory with applications to differential equations, AMS, 1986;

– M. Willem: Minimax Theorems, Birkhauser, 1996.

We want to apply the Mountain Pass Theorem with $X = W_0^{1,2}(\Omega)$ and E the functional defined in (2). We already stated that E is a C^1 functional and satisfies (i) and (ii). It remains to prove that E satisfies $(PS)_c$. This will be accomplished in two steps.

Lemma 15.6 *Any $(PS)_c$-sequence of E is bounded in $W_0^{1,2}(\Omega)$.*

Proof. For every $u \in W_0^{1,2}$ one has that

$$E'(u)[u] - \beta E(u) = \|u\|^2 - \int_\Omega g(x,u)u\,dx - \frac{\beta}{2}\|u\|^2 + \beta\int_\Omega G(x,u)\,dx$$
$$\leq -\left(\frac{\beta}{2} - 1\right)\|u\|^2$$

thanks to (g_2). If (u_n) is a $(PS)_c$-sequence, then

$$\left(\frac{\beta}{2} - 1\right)\|u_n\|^2 \leq \beta E(u_n) + \|E'(u_n)\|\,\|u_n\| \leq C + \delta\|u_n\| \quad \forall n$$

for some $C, \delta > 0$. This implies that (u_n) is bounded, because $\beta > 2$. □

Lemma 15.7 *The functional E satisfies $(PS)_c$.*

Proof. Let $(u_n) \subset W_0^{1,2}$ be a $(PS)_c$-sequence. By Lemma 15.6, (u_n) is bounded in $W_0^{1,2}$. Hence, by well known results (including the Rellich Theorem, and the inverse of the dominated convergence Theorem) it admits a subsequence, still denoted (u_n) such that $u_n \to u$ weakly in $W_0^{1,2}$, strongly in L^{p+1}, and pointwise a.e. and there exists $v \in L^1(\Omega)$ such that $|u_n|^{p+1} \leq v$ a.e. and for all n. (Note that we used the condition $p < \frac{N+2}{N-2}$ ensuring the compact embedding of $W_0^{1,2}$ into L^{p+1}; this step cannot be accomplished if $p = \frac{N+2}{N-2}$). By the dominated convergence Theorem, one obtains that

$$\int_\Omega g(x,u_n)u_n\,dx \to \int_\Omega g(x,u)u\,dx\,, \quad \int_\Omega g(x,u_n)u\,dx \to \int_\Omega g(x,u)u\,dx\,. \quad (5)$$

Moreover

$$E'(u_n)[u_n - u] \to 0 \quad \text{and} \quad E'(u)[u_n - u] = \langle \nabla E(u), u_n - u\rangle \to 0$$

because $\|E'(u_n)\| \to 0$ and $u_n \to u$ weakly. Then

$$(E'(u_n) - E'(u))\,[u_n - u] \to 0$$

On the other hand, using (5)

$$\begin{aligned}(E'(u_n) - E'(u))\,[u_n - u] &= \|u_n - u\|^2 - \int_\Omega [g(x,u_n) - g(x,u)]\,(u_n - u)\,dx \\ &= \|u_n - u\|^2 + o(1)\end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$. Hence $u_n \to u$ strongly in $W_0^{1,2}$. $\square$

Deformations and proof of the Mountain Pass Theorem

We prove the Mountain Pass Theorem by means of the Deformation Lemma stated below. A different strategy, using the Ekeland principle, would be possible (see, e.g., J. Mawhin, M. Willem: Critical Point Theory and Hamiltonian Systems, Springer, 1989). We need some preliminary definition:

Lemma 15.8 (Deformation Lemma) *Let X be a Banach space and let $E\colon X \to \mathbb{R}$ be a functional of class C^1. If $c \in \mathbb{R}$ is not an asymptotic critical level of E, then there exist $\varepsilon_0 > 0$ and for every $\varepsilon \in (0, \varepsilon_0)$ a continuous function $\eta\colon X \to X$ such that:*

$$E(\eta(u)) < c - \varepsilon \text{ if } E(u) < c + \varepsilon \quad \text{and} \quad \eta(u) = u \text{ if } E(u) < c - 2\varepsilon. \tag{6}$$

In other words, the sublevel set $\{E < c + \varepsilon\}$ is changed (in a continuous way) into the sublevel set $\{E < c - \varepsilon\}$, whereas the sublevel set $\{E < c - 2\varepsilon\}$ is unchanged.

Proof of the Mountain Pass Theorem. For every path $\gamma \in \Gamma$, since the mapping $t \mapsto \|\gamma(t)\|$ is continuous, $\|\gamma(0)\| = 0$ and $\|\gamma(1)\| > r$, there exists $t_0 \in (0,1)$ such that $\|\gamma(t_0)\| = r$ and then $\max_{t\in[0,1]} E(\gamma(t)) \geq E(\gamma(t_0)) \geq \alpha$. Therefore $c \geq \alpha$. Arguing by contradiction, suppose that c is not an asymptotic critical level of E. Then we apply the Deformation Lemma taking $2\varepsilon < c - E(0)$ (note that $c - E(0) \geq \alpha - E(0) > 0$). Let $\eta\colon X \to X$ be the continuous mapping satisfying (6). Then take a path $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} E(\gamma(t)) < c + \varepsilon$$

and define

$$\tilde\gamma(t) = \eta(\gamma(t)) \quad \forall t \in [0,1].$$

One has that $\tilde\gamma \in C([0,1]; X)$, $\tilde\gamma(0) = \gamma(0)$ and $\tilde\gamma(1) = \gamma(1)$, because $E(u_1) \leq E(0) < c - 2\varepsilon$, by the choice of ε. Hence $\tilde\gamma \in \Gamma$. Moreover $E(\tilde\gamma(t)) < c - \varepsilon$ for all $t \in [0,1]$. Thus

$$c \leq \max_{t\in[0,1]} E(\tilde\gamma(t)) < c - \varepsilon$$

a contradiction. Therefore c has to be an asymptotic critical level of E. Finally, assume that, in addition, E satisfies $(PS)_c$. Having proved that c is an asymptotic critical level of E, there exists a sequence $(u_n) \subset X$ such that

$E(u_n) \to c$ and $\|E'(u_n)\| \to 0$. Since $(PS)_c$ holds, (u_n) admits a subsequence, still denoted (u_n), strongly converging to some $u \in X$. Since E is of class C^1, $E(u) = \lim E(u_n) = c$ and $\|E'(u)\| = \lim \|E'(u_n)\| = 0$, namely c is a critical level. $\square$

Proof of the Deformation Lemma. We discuss the proof in a simplified case, assuming X Hilbert space and $E \in C^{1,1}(X, \mathbb{R})$, namely E differentiable with $E' \colon X' \to \mathbb{R}$ locally Lipschitz-continuous. We construct the function $\eta \colon X \to X$ by solving a Cauchy problem in X. First of all, we observe that since c is not an asymptotic critical level, there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that

$$\|E'(u)\| \geq \delta \quad \forall u \in \{c - \varepsilon_0 < E < c + \varepsilon_0\}.$$

In order to show the above statement, argue by contradiction. Fix $\varepsilon \in (0, \varepsilon_0)$ and define

$$F(u) = -\frac{\nabla E(u)}{1 + \|\nabla E(u)\|} \psi(E(u))$$

where $\nabla E(u)$ is the gradient of E at u, namely the representative of $E'(u)$ in X given by the Riesz Theorem (here we use the fact that X is a Hilbert space), and $\psi \colon \mathbb{R} \to \mathbb{R}$ is a C^∞ mapping such that

$$\begin{cases} 0 \leq \psi(s) \leq 1 & \forall s \in \mathbb{R} \\ \psi(s) = 0 & \text{if } s \leq c - 2\varepsilon \\ \psi(s) = 1 & \text{if } s \geq c - \varepsilon. \end{cases}$$

Then, for every fixed $u \in X$ consider the Cauchy problem

$$\begin{cases} \phi' = F(\phi) \\ \phi(0) = u\,. \end{cases} \tag{7}$$

The vector field $F \colon X \to X$ turns out to be locally Lipschitz continuous. This ensures existence and uniqueness of a local solution $\phi(t, u)$ to the Cauchy problem (7) as well as continuous dependence with respect to u. Moreover $\|F(u)\| \leq 1$ for every $u \in X$. This implies that any solution is global, namely it is defined for every $t \in \mathbb{R}$. In addition, we compute

$$\frac{d}{dt} E(\phi(t)) = E'(\phi(t))[\phi'(t)] = -\frac{\|\nabla E(\phi(t))\|^2}{1 + \|\nabla E(\phi(t))\|} \psi(E(\phi(t))) \leq 0$$

which means that the functional is decreasing along the flow ϕ. In particular, if one starts from a point $u \in X$ where $E(u) < c + \varepsilon$ and $E(\phi(t, u)) \geq c - \varepsilon$ $\forall t \in [0, T]$, then

$$\begin{aligned} c - \varepsilon \leq E(\phi(T, u)) &= E(\phi(0, u)) + \int_0^T \frac{d}{dt} E(\phi(t))\, dt \\ &= E(u) - \int_0^T \frac{\|\nabla E(\phi(t))\|^2}{1 + \|\nabla E(\phi(t))\|} \psi(E(\phi(t)))\, dt < c + \varepsilon - \frac{\delta^2 T}{1 + \delta} \end{aligned}$$

because $\psi(E(\phi(t))) = 1$ and $\|\nabla E(\phi(t))\| = \|E'(\phi(t))\| \geq \delta$ for all $t \in [0, T]$. In particular it must be $T < \frac{2\varepsilon\delta^2}{1+\delta}$. In other words, setting

$$T_\varepsilon = \frac{2\varepsilon\delta^2}{1+\delta},$$

if $E(u) < c + \varepsilon$, then $E(\phi(T_\epsilon, u)) < c - \varepsilon$. Finally, observe that if $E(u) \leq c - 2\varepsilon$ then $F(u) = 0$ and then $\phi(t, u) = u \ \forall t \in \mathbb{R}$. In conclusion, the mapping

$$\eta(u) = \phi(T_\varepsilon, u)$$

satisfies the required properties (6). $\square$