16 Standing waves of nonlinear Schrödinger equations

In Quantum Mechanics the evolution of a (non-relativistic) single particle of unit mass in a N-dimensional space is described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + V(x)\psi$$

where i is the imaginary unit, $\hbar$ is the Planck constant, $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, Δ denotes the Laplace operator (with respect to x), $\psi = \psi(t,x)$ is a complex-valued function, called *wave function*, and $V(x)$ is the potential energy. The same equation models a system of n particles; in such a case $x = (x_1, ..., x_n) \in (\mathbb{R}^N)^n$.

In the theory of Bose-Einstein condensates, one considers a system of a very large number of particles and takes two limits, sending the number of particles n to infinity and the interaction scale to zero. Moreover one models the interaction of one particle with all the other particles in the so-called mean field approximation. From symmetry, all particles turn out to have the same state function $\rho(t,x) = \psi(t,x)\overline{\psi(t,x)}$ (with $x \in \mathbb{R}^N$) and the one-particle wave function is governed by a non-linear equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + V(x)\psi + \mu|\psi|^2\psi$$

where μ is a real constant, $V(x)$ is the external potential and the non linearity, given by cubic term $|\psi|^2\psi$, comes from the interaction between particles. More generally, the *nonlinear Schrödinger equation* takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + V(x)\psi + \mu|\psi|^{2\sigma}\psi \tag{16.1}$$

with $\sigma > 0$. A *stationary wave* of (16.1) is a solution of (16.1) of the form

$$\psi(t,x) = \exp(i\omega\hbar^{-1}t)u(x) \text{ with } \omega \in \mathbb{R} \text{ and } u(x) \in \mathbb{R},\ u > 0.$$

We will consider the case in which $\mu < 0$. Thus, after rescaling and renaming terms in (16.1), looking for stationary waves of NLS equations is equivalent to finding solutions to

$$\begin{cases} -\Delta u + q(x)u = u^p \text{ in } \mathbb{R}^N \\ u \in W^{1,2}(\mathbb{R}^N),\ u > 0 \text{ in } \mathbb{R}^N \end{cases} \tag{16.2}$$

where the condition $u \in W^{1,2}(\mathbb{R}^N)$ is added in order to obtain bound states, namely solutions with finite energy. In case $q(x)$ is a positive constant, the equation in (16.2) is called also scalar field equation and appears in some studies in nuclear physics.

Usually the potential energy $V(x)$ in (16.1) is a function bounded from below. Moreover the relationship between $V(x)$ in (16.1) and $q(x)$ in (16.2) is of the form $q(x) = \hbar^{-2}(V(x) + \omega)$. Hence, taking ω large enough, we can assume

$$\inf_{x\in\mathbb{R}^N} q(x) > 0.$$

In fact, in this study, we restrict to ourselves to coefficients $q(x)$ which are bounded also from above. As far as concerns the nonlinearity, we assume $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, whereas $p > 1$ if $N = 1, 2$. Under these assumptions on $q(x)$ and p, the *energy functional*

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + q(x)u^2)\,dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1}\,dx$$

is well defined and Fréchet-differentiable at every $u \in W^{1,2}(\mathbb{R}^N)$ and critical points of E are weak solutions of

$$\begin{cases} -\Delta u + q(x)u = |u|^{p-1}u \ \text{ in } \mathbb{R}^N \\ u \in W^{1,2}(\mathbb{R}^N) \end{cases} \tag{16.3}$$

and viceversa.

A nontrivial (i.e., non zero) solution u_0 of (16.3) is called *ground state* if has least energy among all the non zero, weak solutions of (16.3), namely $E(u_0) \leq E(u)$ for every solution u of (16.3).

Remark 16.1. *Recall that $W^{1,2}(\mathbb{R}^N)$ is the space of functions $u \in L^2(\mathbb{R}^N)$ with weak derivatives $\frac{\partial u}{\partial x_j} \in L^2(\mathbb{R}^N)$ $(j = 1, ..., N)$. It is also denoted $H^1(\mathbb{R}^N)$ and is a Hilbert space endowed with the inner product*

$$\langle u, v\rangle_{W^{1,2}} = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv)\,dx\,.$$

The subspace $C_c^\infty(\mathbb{R}^N)$ of smooth functions with compact support turns out to be dense in $W^{1,2}(\mathbb{R}^N)$. Moreover $W^{1,2}(\mathbb{R}^N)$ is embedded into $L^q(\mathbb{R}^N)$ for all $q \in [2, \frac{2N}{N-2}]$ in dimension $N \geq 3$, and for all $q \geq 2$ in dimension $N = 1, 2$. Differently from the case of bounded domain, the embedding $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is never compact. For example, fixing $u \in W^{1,2}(\mathbb{R}^N)$, $u \neq 0$, taking a sequence $(x_n) \subset \mathbb{R}^N$ with $|x_n| \to \infty$, and setting $u_n(x) = u(x - x_n)$, the sequence (u_n) converges weakly to 0 in $W^{1,2}(\mathbb{R}^N)$ and also in $L^q(\mathbb{R}^N)$, but not strongly. In fact $\|u_n\|_{W^{1,2}} = \|u\|_{W^{1,2}}$ and $\|u_n\|_{L^q} = \|u\|_{L^q}$.

A strategy to get existence of solutions to (16.2) under the above assumptions is the following: set

$$F(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + q(x)u^2)\,dx \quad \forall u \in W^{1,2}(\mathbb{R}^N)$$
$$M = \{u \in W^{1,2}(\mathbb{R}^N) \mid \|u\|_{L^{p+1}} = 1\}$$

and study the minimization problem

$$\inf_{u\in M} F(u)\,. \tag{16.4}$$

Suppose, for a moment, that

$$\exists \overline{u}\in M,\ \overline{u}\geq 0\ /\ F(\overline{u})=\inf_{u\in M} F(u). \tag{16.5}$$

Now, consider the functional

$$\widetilde{F}(u)=\frac{F(u)}{\|u\|^2_{L^{p+1}}}\quad \forall u\in W^{1,2}(\mathbb{R}^N)\setminus\{0\}.$$

Since $p\leq \frac{N+2}{N-2}$ as $N\geq 3$, the functional $\widetilde{F}$ is well defined and Fréchet-differentiable at every $u\in W^{1,2}(\mathbb{R}^N)\setminus\{0\}$. In particular, for any $h\in W^{1,2}(\mathbb{R}^N)$

$$\widetilde{F}'(u)[h]=\frac{2\|u\|^2_{L^{p+1}}\int_{\mathbb{R}^N}(\nabla u\cdot\nabla h+q(x)uh)\,dx-2F(u)\int_{\mathbb{R}^N}|u|^{p-1}uh\,dx}{\|u\|^4_{L^{p+1}}}.$$

Since $W^{1,2}(\mathbb{R}^N)\setminus\{0\}$ is an open subset of the space $W^{1,2}(\mathbb{R}^N)$, and $\widetilde{F}(\overline{u})=\min_{u\in W^{1,2}(\mathbb{R}^N)\setminus\{0\}}\widetilde{F}(u)$, by Fermat's Theorem, $\widetilde{F}'(\overline{u})=0$ and then, as $\overline{u}\in M$ and $\overline{u}\geq 0$,

$$\int_{\mathbb{R}^N}(\nabla\overline{u}\cdot\nabla h+q(x)\overline{u}h)\,dx=\mu\int_{\mathbb{R}^N}\overline{u}^p h\,dx\quad\forall h\in W^{1,2}(\mathbb{R}^N)$$

namely $\overline{u}$ is a weak, non zero, solution to

$$\begin{cases} -\Delta u+q(x)u=\mu u^p\ \text{ in }\mathbb{R}^N\\ u\in W^{1,2}(\mathbb{R}^N),\ u\geq 0\ \text{ in }\mathbb{R}^N\end{cases}$$

Then one can complete the argument as in the case of the Dirichlet problem

$$\begin{cases} -\Delta u+q(x)u=u^p\ \text{ in }\Omega\\ u\in W^{1,2}_0(\Omega),\ u>0\ \text{ in }\Omega\end{cases} \tag{16.6}$$

on a bounded domain Ω in $\mathbb{R}^N$. More precisely, one can check that $u=\mu^{\frac{1}{p-1}}\overline{u}$ is a weak, possibly non negative, solution to problem (16.2). As soon as q is regular enough, for example $q\in C^{0,\alpha}_{loc}(\mathbb{R}^N)$, regularity theory applies and one infers that u is in fact a classical solution. Finally, by the maximum principle, one also deduces that $u>0$ on $\mathbb{R}^N$. Furthermore a solution found as before, by solving the minimization problem (16.4) turns out to be a least energy solution.

Proposition 16.2. *If (16.5) holds true, then $u=\mu^{\frac{1}{p-1}}\overline{u}$ is a ground state for problem (16.3).*

Proof. If $u \in W^{1,2}(\mathbb{R}^N)$ is a weak solution to (16.2) then $E'(u)[u] = 0$, namely $F(u) = \|u\|^{p+1}_{L^{p+1}}$. Then $\widetilde{F}(u) = F(u)^{\frac{p}{p+1}}$ and $E(u) = \big(\frac{1}{2} - \frac{1}{p+1}\big)F(u) = \big(\frac{1}{2} - \frac{1}{p+1}\big)\widetilde{F}(u)^{\frac{p+1}{p}}$. Since $\widetilde{F}(\overline{u}) \leq \widetilde{F}(u)$, the conclusion follows. □

Hence the key step is given by (16.5). Here a substantial difference with respect to the case of the Dirichlet problem (16.6) on a bounded domain appears. Indeed, since the domain is the whole space $\mathbb{R}^N$, the minimization set M is not weakly closed in $W^{1,2}(\mathbb{R}^N)$ (see Remark 16.1) and then the weak limit of a minimizing sequence for problem (16.4) does not necessarily belong to M.

In fact the existence of a minimizer for problem (16.4), and actually the existence of a solution for problem (16.2) is strongly affected by the behaviour of the coefficient $q(x)$. For example, one can show that if q depends just on one variable in a strictly monotone fashion, then (16.2) has no solution.

1. The case of constant coefficient

Here we study the minimization problem when q is a positive constant and we show:

Theorem 16.3. *Assume $q(x) = q_0 > 0$ for all $x \in \mathbb{R}^N$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, or $p > 1$ if $N = 1, 2$. Then there exists a radial mapping $\overline{u} \in M$, radially decreasing, non negative, and such that $F(\overline{u}) = \inf_{u \in M} F(u)$.*

By radial function we mean a mapping $u\colon \mathbb{R}^N \to \mathbb{R}$ such that $u(x) = v(|x|)$ for a.e. $x \in \mathbb{R}^N$ and $v\colon [0, \infty) \to \mathbb{R}$. Moreover u is radially decreasing if v is so.

As a consequence of Theorem 16.3, following the argument previously described, one can plainly obtain that:

Theorem 16.4. *Under the assumptions of Theorem 16.3, problem (16.2) admits a radial, radially decreasing, classical solution.*

The proof of Theorem 16.3 rests on the possibility to substitute any minimizing sequence with another one, made by radially symmetric and non negative functions, via Schwarz symmetrization. This can be accomplished because we are dealing with the case of constant coefficient. Another key tool is given by the following compactness result. Let

$$W^{1,2}_r(\mathbb{R}^N) = \{u \in W^{1,2}(\mathbb{R}^N) \mid u \text{ radial}\}.$$

Theorem 16.5 (Strauss radial lemma, 1977)**.** *In dimension $N \geq 2$, for $q \in (2, \frac{2N}{N-2})$ the embedding $W^{1,2}_r(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact.*

The basic step is the next estimate:

Lemma 16.6. *There is a constant $C_N > 0$, depending just on the dimension N, such that for every $u \in W^{1,2}_r(\mathbb{R}^N)$*

$$|u(x)| \le C_N |x|^{-\frac{N-1}{2}} \|u\|_{W^{1,2}} \quad \textit{a.e. on } \mathbb{R}^N. \tag{16.7}$$

Proof. Take $u \in W^{1,2}_r \cap C^\infty_c(\mathbb{R}^N)$. Then $u(x) = v(|x|)$ and

$$\begin{aligned} |x|^{N-1} u(x)^2 &= -\int_{|x|}^\infty \left(r^{N-1} v(r)^2\right)' dr \le -2 \int_{|x|}^\infty r^{N-1} v(r) v'(r)\, dr \\ &\le \int_{|x|}^\infty r^{N-1} (v(r)^2 + v'(r)^2)\, dr \le \sigma_N^{-1} \|u\|^2_{W^{1,2}} \end{aligned}$$

where σ_N is the $(N-1)$-dimensional area of the sphere $\{x \in \mathbb{R}^N \mid |x| = 1\}$. Hence (16.7) holds true with $C_N = \sigma_N^{-1/2}$ and $u \in W^{1,2}_r \cap C^\infty_c(\mathbb{R}^N)$. By a density argument, one obtains the same estimate also for every $u \in W^{1,2}_r(\mathbb{R}^N)$. $\square$

Proof of the Strauss radial lemma. Let (u_n) be a sequence in $W^{1,2}_r(\mathbb{R}^N)$ which converges to 0 weakly in $W^{1,2}$. Then $\sup_n \|u_n\|_{H_1} < \infty$ and therefore, by Lemma 16.6, there is $C > 0$ independent of n such that

$$|u_n(x)| \le C |x|^{-\frac{N-1}{2}} \quad \text{a.e. on } \mathbb{R}^N \text{ and } \forall n. \tag{16.8}$$

Let $q \in (2, \frac{2N}{N-2})$ and fix $\varepsilon > 0$. By (16.8) there exists $R_\varepsilon > 0$ such that

$$|u_n(x)|^{q-2} \le \varepsilon \quad \text{for a.e. } |x| \ge R_\varepsilon \text{ and } \forall n.$$

(Here one uses $q > 2$ and also $N \ge 2$). Since the embedding $W^{1,2}(B_{R_\varepsilon}) \hookrightarrow L^q(B_{R_\varepsilon})$ is compact (by Rellich's Theorem and since $q < \frac{2N}{N-2}$), one has that

$$\int_{B_{R_\varepsilon}} |u_n|^q dx \to 0\,.$$

Moreover, one has

$$\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |u_n|^q dx \le \varepsilon \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |u_n|^2 dx \le \varepsilon \|u_n\|^2_{W^{1,2}} \le C\varepsilon\,.$$

Then

$$\limsup \int_{\mathbb{R}^N} |u_n|^q dx \le \limsup \int_{B_{R_\varepsilon}} |u_n|^q dx + \limsup \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |u_n|^q dx \le C\varepsilon$$

and by the arbitrariness of ε one infers that $\|u_n\|_{L^q} \to 0$. Now let (u_n) be a sequence in $W_r^{1,2}(\mathbb{R}^N)$ which converges to u weakly in $W^{1,2}$. Then $v_n = u_n - u \to 0$ weakly in $W^{1,2}$ and using the previous part, one concludes that $\|v_n\|_{L^q} \to 0$, namely $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$. □

Proof of Theorem 16.3. Set $\mu = \inf_{u\in M} F(u)$, where $F(u) = \int_{\mathbb{R}^N}(|\nabla u|^2 + q_0u^2)\,dx$. Let $(u_n) \subset M$ be such that $F(u_n) \to \mu$. For every n let u_n^* be the spherical decreasing rearrangement of u_n. We know that $u_n^* \in W_r^{1,2}(\mathbb{R}^n)$ satisfies $\|u_n^*\|_{L^q} = \|u_n\|_{L^q}$ for all $q \in [2, \frac{2N}{N-2}]$ (if $N \geq 3$, otherwise for all $q \geq 2$) and $\|\nabla u_n^*\|_{L^2} \leq \|\nabla u_n\|_{L^2}$. From this it follows that also $(u_n^*) \subset M$ and $F(u_n^*) \to \mu$. One can easily check that (u_n^*) is bounded in $W^{1,2}(\mathbb{R}^N)$, hence, passing to a subsequence, $u_n^* \to \overline{u}$ weakly in $W^{1,2}(\mathbb{R}^N)$. By the Strauss radial lemma, $u_n^* \to \overline{u}$ strongly in $L^{p+1}(\mathbb{R}^N)$ (here one uses the strict inequality $p < \frac{N+2}{N-2}$). Therefore, in particular $\|\overline{u}\|_{L^{p+1}} = 1$, $\overline{u}$ is radial, radially decreasing, and non negative (because each u_n^* is so, and, for a subsequence $u_n^* \to \overline{u}$ pointwise a.e.). Moreover, by weak lower semicontinuity, $F(\overline{u}) \leq \liminf F(u_n^*)$. Since $\overline{u} \in M$ and $\liminf F(u_n^*) = \mu$, one deduces that $F(\overline{u}) = \mu$. Thus Theorem 16.3 is proved. □

Remark 16.7. *(1) H. Berestycki and P.L. Lions in 1983 proved that problem*

$$\begin{cases} -\Delta u + u = u^p \ \text{ on } \mathbb{R}^N \\ u \in L^{p+1}(\mathbb{R}^N),\ u > 0 \ \text{ in } \mathbb{R}^N \end{cases} \tag{16.9}$$

has no solution for $p \geq \frac{N+2}{N-2}$ (if $N \geq 3$).

(2) In 1981, B. Gidas, W.-M. Ni, and L. Nirenberg proved that if u is a positive solution to $-\Delta u + u = u^p$ on $\mathbb{R}^N$ and $u(x) \to 0$ as $|x| \to \infty$, then $u(x)$ is spherically symmetric about some point $x_0 \in \mathbb{R}^N$, $\frac{\partial u}{\partial r} < 0$ for $r > 0$, where r is the radial coordinate about x_0. Furthermore, there exists $\alpha > 0$ such that $u(r) \sim \alpha r^{-\frac{N-1}{2}} e^{-r}$ as $r \to \infty$.

(3) M.K. Kwong in 1989 established the uniqueness of the positive, radially symmetric solution to (16.9).

2. The case q asymptotic to a constant at infinity

Now we study the minimization problem (16.4) in the case $q \in L^\infty(\mathbb{R}^N)$ with $\inf_{x\in\mathbb{R}^N} q(x) > 0$ and $\lim_{|x|\to\infty} q(x) = q_\infty \in (0, \infty)$. It is convenient to introduce the values

$$\mu = \inf_{u\in M} F(u) \quad \text{and} \quad \mu_\infty = \inf_{u\in M} F_\infty(u)$$

where $F_\infty(u) = \int_{\mathbb{R}^N}(|\nabla u|^2 + q_\infty u^2)\,dx$.

Theorem 16.8. *One has that $\mu \le \mu_\infty$. Moreover, if $\mu < \mu_\infty$ then there exists $\overline{u} \in M$ with $\overline{u} \ge 0$ and $F(\overline{u}) = \mu$.*

Proof. We know that there exists $\overline{u} \in M$ with $\overline{u} > 0$ and $F_\infty(\overline{u}) = \mu_\infty$. Take a sequence $(x_n) \subset \mathbb{R}^N$ such that $|x_n| \to \infty$ and set $u_n(x) = \overline{u}(x - x_n)$. Then $u_n \in M$ and $F(u_n) \to F_\infty(\overline{u})$. Indeed $\|\nabla u_n\|_{L^2} = \|\nabla \overline{u}\|_{L^2}$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} q(x) u_n^2 \, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} q(x + x_n) \overline{u}^2 \, dx = \int_{\mathbb{R}^N} q_\infty \overline{u}^2 \, dx,$$

by the Lebesgue theorem. Hence $\mu \le \lim F(u_n) = F_\infty(\overline{u}) = \mu_\infty$. The proof of the second part of the Theorem is based on the following result.

Lemma 16.9 (H. Brezis, E. Lieb, 1983)**.** *Let Ω be a domain in $\mathbb{R}^N$ and let $(u_n) \subset L^p(\Omega)$, $1 \le p < \infty$. If (u_n) is bounded in $L^p(\Omega)$ and $u_n \to u$ a.e. in Ω, then*

$$\lim_{n\to\infty} \left(\|u_n\|_{L^p}^p - \|u_n - u\|_{L^p}^p \right) = \|u\|_{L^p}^p.$$

Let us complete the proof of Theorem 16.8. Let $(u_n) \subset M$ be such that $F(u_n) \to \mu$. Since $\inf_{x\in\mathbb{R}^N} q(x) > 0$, the sequence (u_n) is bounded in $W^{1,2}(\mathbb{R}^N)$. Hence, passing to a subsequence, we can assume that $u_n \to u$ weakly in $W^{1,2}(\mathbb{R}^N)$ and pointwise a.e. in $\mathbb{R}^N$. Then

$$F(u_n) - F(u_n - u) = 2 \int_{\mathbb{R}^N} (\nabla u_n \cdot \nabla u + q(x) u_n u) \, dx - F(u) \to F(u) \ \text{ as } n \to \infty.$$

Since (u_n) is bounded also in L^{p+1}, by Lemma 16.9 one has

$$\|u_n\|_{L^{p+1}}^{p+1} - \|u_n - u\|_{L^{p+1}}^{p+1} \to \|u\|_{L^{p+1}}^{p+1} \ \text{ as } n \to \infty.$$

Set $\lambda = \|u\|_{L^{p+1}}^{p+1}$ and observe that by the Fatou lemma $\lambda \in [0, 1]$. Then $\|u_n - u\|_{L^{p+1}} \to (1 - \lambda)^{\frac{1}{p+1}}$. Moreover we claim that

$$F(u_n - u) - F_\infty(u_n - u) \to 0 \ \text{ as } n \to \infty. \tag{16.10}$$

Indeed, fix $\varepsilon > 0$. Since $\lim_{|x|\to\infty} q(x) = q_\infty$, there exists $R_\varepsilon > 0$ such that $|q(x) - q_\infty| < \varepsilon$ as $|x| \ge R_\varepsilon$. We have that

$$|F(u_n - u) - F_\infty(u_n - u)| = \int_{\mathbb{R}^N} |q(x) - q_\infty| \, (u_n - u)^2 \, dx$$

Since $u_n \to u$ weakly in $W^{1,2}(B_{R_\varepsilon})$, by the Rellich theorem, $u_n \to u$ strongly in $L^2(B_{R_\varepsilon})$. Therefore

$$\lim_{n\to\infty} \int_{B_{R_\varepsilon}} |q(x) - q_\infty| (u_n - u)^2 \, dx = 0.$$

Moreover

$$\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} |q(x)-q_\infty|(u_n-u)^2\,dx \le \varepsilon \int_{\mathbb{R}^N} (u_n-u)^2\,dx \le C\varepsilon$$

for some $C>0$ independent of n. Then

$$\limsup \int_{\mathbb{R}^N} |q(x)-q_\infty|(u_n-u)^2\,dx \le C\varepsilon\,.$$

By the arbitrariness of $\varepsilon>0$ we obtain that (16.10) holds true. Now we observe that

$$F(u) \ge \mu \|u\|^2_{L^{p+1}} = \lambda^{\frac{2}{p+1}}\mu$$
$$F_\infty(u_n-u) \ge \mu_\infty \|u_n-u|^2_{L^{p+1}} = (1-\lambda)^{\frac{2}{p+1}}\mu_\infty + o(1)$$

and then we can estimate

$$\begin{aligned}\mu &= F(u_n)+o(1) = F(u_n-u)+F(u)+o(1)\\ &= F_\infty(u_n-u)+F(u)+o(1) \ge (1-\lambda)^{\frac{2}{p+1}}\mu_\infty + \lambda^{\frac{2}{p+1}}\mu + o(1)\,.\end{aligned} \tag{16.11}$$

Since $\mu_\infty \ge \mu > 0$ (thanks to the embedding $W^{1,2}(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N)$), we obtain

$$1 \ge (1-\lambda)^{\frac{2}{p+1}} + \lambda^{\frac{2}{p+1}}\,. \tag{16.12}$$

Recalling that $\lambda \in [0,1]$ and observing that the function $t \mapsto t^{\frac{2}{p+1}}$ is concave, (16.12) implies $\lambda = 0$ or $\lambda = 1$. The case $\lambda = 0$ cannot occur, because otherwise, from (16.11) it follows $\mu \ge \mu_\infty$, contrary to the assumption $\mu < \mu_\infty$. Hence $\lambda = 1$ is the only possible case. This means that $u \in M$, and then $F(u) \ge \mu$. On the other hand, by weak lower semicontinuity, $F(u) \le \liminf F(u_n) = \mu$. In conclusion $F(u) = \mu$, namely u is a minimum point for problem (16.4). Since $|u| \in M$ and $F(|u|) \le F(u)$, we obtain that $|u|$ is a non negative minimum point and this completes the proof. □

According to the proof of Theorem 16.8, one can see that there are two possibilities: either $\mu = \mu_\infty$ and in this case there exist noncompact minimimizing sequences, or $\mu < \mu_\infty$ and in this case in fact all minimizing sequences admit strongly convergent subsequences (this follows from the fact that F is equivalent to the standard norm of $W^{1,2}(\mathbb{R}^N)$ and we proved that $F(u_n) \to F(u)$ and $u_n \to u$ weakly in $W^{1,2}(\mathbb{R}^N)$).

Theorem 16.10. *Assume $q \in L^\infty(\mathbb{R}^N)$ with $\inf_{x\in\mathbb{R}^N} q(x) > 0$ and $\lim_{|x|\to\infty} q(x) = q_\infty \in (0,\infty)$. If $q(x) \le q_\infty$ for every $x \in \mathbb{R}^N$, then there exists $\overline{u} \in M$ with $\overline{u} \ge 0$ and $F(\overline{u}) = \inf_{u\in M} F(u)$. Moreover, if in addition $q \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$, then problem (16.1) admits a positive, classical solution.*

Proof. If $q(x) = q_\infty$ for a.e. $x \in \mathbb{R}^N$ then we already proved the existence of a minimizer. It remains the case $q(x) < q_\infty$ for $x \in E$ where E is a set of positive measure. Take the minimizer u_∞ for μ_∞. Since $\overline{u} > 0$, we have that

$$F_\infty(u_\infty) - F(u_\infty) \geq \int_E (q_\infty - q(x)) u_\infty^2 \, dx > 0$$

and then $\mu \leq F(u_\infty) < F_\infty(u_\infty) = \mu_\infty$. Hence Theorem 16.8 can be applied and the conclusion follows. □

Remark 16.11. *When $q(x) \geq q_\infty$ for every $x \in \mathbb{R}^N$ and $q(x) > q_\infty$ on a set of positive measure, then the minimization problem (16.4) has no solution. In fact, suppose by contradiction that there is $\overline{u} \in M$ such that $F(\overline{u}) = \mu$. Taking the absolute value, we can also assume $\overline{u} \geq 0$. Moreover one has $\mu = F(\overline{u}) \geq F_\infty(\overline{u}) \geq \mu_\infty$ and then $\overline{u}$ is a minimum point also for F_∞. Therefore $\overline{u} > 0$ everywhere and up to a multiplicative constant, it solves both $\Delta u - q_\infty u + u^p = 0$ and $\Delta u - q(x)u + u^p = 0$. By subtracting these equations, since $\overline{u} > 0$, one obtains $q(x) = q_\infty$ for a.e. $x \in \mathbb{R}^N$, which is false. Therefore if $q(x) \geq q_\infty$ for every $x \in \mathbb{R}^N$ and $q(x) > q_\infty$ on a set of positive measure, then every minimizing sequence for problem (16.4) cannot converge strongly. Clearly this does not mean that problem (16.2) has no solution. In fact, as proved by A. Bahri and Y.Y. Li in 1990 for a similar problem, if $q(x)$ tends to q_∞ as $|x| \to \infty$ with some decay rate, then one can find a positive critical point of the energy functional, namely a solution to (16.2), even if the infimum of F in M is not attained.*

According to the previus discussion, if $k \in C_c^\infty(\mathbb{R}^N)$ is positive on its support, then the infimum problem (16.4) with $q(x) = 1 + k(x)$ has a minimizer. In fact the set of mimimum points is compact. Taking $q(x) = 1 + k(x)$, problem (16.4) has no solution, whereas for $q(x) \equiv 1$ the set of minimizers is given by $S = \{u(\cdot - x_0) \mid x_0 \in \mathbb{R}^N\}$ where u is the (unique) radial, positive minimizer. Hence, small perturbations of the case of constant coefficients have drastic effects on the problem of the existence of minimizers for (16.4).

Proof of Lemma 16.9. We use the following inequality:

$$(*) \qquad \forall \varepsilon > 0 \; \exists C_\varepsilon \; / \; \big||a+b|^p - |a|^p - |b|^p\big| \leq \varepsilon |a|^p + C_\varepsilon |b|^p \; \forall a, b \in \mathbb{R}.$$

More precisely we apply $(*)$ with $a = u_n(x) - u(x)$ and $b = u(x)$, obtaining that

$$\big||u_n(x)|^p - |u_n(x) - u(x)|^p - |u(x)|^p\big| \leq \varepsilon |u_n(x) - u(x)|^p + C_\varepsilon |u(x)|^p.$$

Hence, setting

$$f_n(x) = \big||u_n(x)|^p - |u_n(x) - u(x)|^p - |u(x)|^p\big| - \varepsilon |u_n(x) - u(x)|^p, \; g_n = [f_n]_+,$$

we have that $g_n \to 0$ a.e. in Ω and $0 \le g_n \le C_\varepsilon |u|^p$. Observe that $|u|^p \in L^1(\Omega)$. Indeed by hupothesis there exists $C > 0$ such that

$$\int_\Omega |u_n|^p \, dx \le C \quad \forall n$$

and, by the Fatou lemma,

$$\int_\Omega |u|^p \, dx \le \liminf \int_\Omega |u_n|^p \, dx \le C.$$

Hence we can apply the Lebesgue dominated convergence theorem and infer that

$$\int_\Omega g_n \, dx \to 0 \, .$$

Moreover

$$\int_\Omega \big| |u_n|^p - |u_n - u|^p - |u|^p \big| \, dx \le \int_\Omega \big(f_n + \varepsilon |u_n - u|^p \big) \, dx \le \int_\Omega g_n \, dx + \varepsilon \|u_n - u\|_p^p$$

and the sequence $(u_n - u)$ is bounded in $L^p(\Omega)$. Hence

$$\limsup \int_\Omega \big| |u_n|^p - |u_n - u|^p - |u|^p \big| \, dx \le \varepsilon C_1$$

for some constant $C_1 > 0$ independent of n and ε. Then the conclusion follows from the arbitrariness of $\varepsilon > 0$.

Proof of $(*)$: if $b = 0$ it is obvious. If $b \neq 0$, $(*)$ is equivalent to

$$\forall \varepsilon > 0 \; \exists C_\varepsilon \; / \; \big| |t+1|^p - |t|^p - 1 \big| \le \varepsilon |t|^p + C_\varepsilon \; \forall t \in \mathbb{R}$$

namely the function $F(t) = \big| |t+1|^p - |t|^p \big| - \varepsilon |t|^p$ is bounded from above. This is true because F is continuous and for $\varepsilon > 0$

$$\lim_{|t| \to \infty} F(t) = \lim_{|t| \to \infty} |t|^p \left(\big| |1 + 1/t|^p - 1 \big| - \varepsilon \right) = -\infty. \; \square$$