The Hilbert-Schmidt Theorem

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Reference: Kolmogorov - Fomin: Elements of the theory of functions and Functional Analysis

H real or complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$.
In case of complex space, recall that $\langle u, v \rangle = \overline{\langle v, u \rangle}$.
$T : H \to H$ linear operator

def 1 T is compact if $u_n \xrightarrow{w} u \Longrightarrow Tu_n \xrightarrow{s} Tu$

Rmk 1 T compact $\Rightarrow$ T continuous.

def 2 T is self-adjoint if $\langle Tu, v \rangle = \langle u, Tv \rangle \quad \forall u, v \in H$.

Rmk 2 If T is self-adjoint then $\langle Tu, u \rangle \in \mathbb{R} \quad \forall u \in H$, because
$\langle Tu, u \rangle = \langle u, Tu \rangle = \overline{\langle Tu, u \rangle}$.

def 3 A number $\lambda \in \mathbb{C}$ is an eigenvalue of T if $\exists\, u \in H$ $u \neq 0$ s.t. $Tu = \lambda u$. Such u is called eigenvector (or eigenfunction when H is a space of functions) corresponding to the eigenvalue λ. Moreover the set $\{u \in H : Tu = \lambda u\}$ is a linear subspace of H called eigenspace corresponding to λ.

Rmk 3 T self-adjoint $\Longrightarrow$ only real eigenvalues

$$\left.\begin{array}{l} Tu = \lambda u \\ u \neq 0 \end{array}\right. \Rightarrow \langle Tu, u \rangle = \lambda \|u\|^2 \Longrightarrow \lambda \in \mathbb{R} \quad \text{(by Rmk 2)}.$$

Theorem Let H be a Hilbert space and let $T : H \to H$ be a linear, compact, self-adjoint operator. Then $\exists$ at most countable sets $(\lambda_n^+)_{n \in I^+}, (\lambda_n^-)_{n \in I^-} \subset \mathbb{R}$, $(v_n^+)_{n \in I^+}$, $(v_n^-)_{n \in I^-} \subset H$ s.t.

1) $\lambda_1^- \leq \lambda_2^- \leq \dots < 0 < \dots \leq \lambda_2^+ \leq \lambda_1^+$
2) $\lambda_n^\pm$ are eigenvalues and $v_n^\pm$ corresponding eigenvectors
3) the $v_n^\pm$'s are orthonormal vectors
4) if (λ_n^+) is infinite then $\lambda_n^+ \to 0$ (the same for (λ_n^-))

5) if μ is a nonzero eigenvalue then $\mu \in \{\lambda_n^+\} \cup \{\lambda_n^-\}$

6) $H = H_- \oplus H_0 \oplus H_+$ where $H_\pm = \overline{\text{span}\{v_n^\pm\}}$, $H_0 = \ker T$.

Lemma Set $\lambda = \sup_{\substack{u \in H \\ u \neq 0}} \frac{\langle Tu, u\rangle}{\|u\|^2}$ - if $\lambda > 0$ then λ is an eigenvalue.

Proof Note that $\lambda \leq \|T\|$ (because $\langle Tu, u\rangle \leq \|Tu\|\,\|u\| \leq \|T\|\,\|u\|^2$). Set $F(u) = \frac{\langle Tu, u\rangle}{\|u\|^2}$ $\forall u \in H \setminus \{0\}$.

Let $(u_n) \subset H \setminus \{0\}$ be s.t. $F(u_n) \longrightarrow \lambda$. Set $v_n = \frac{u_n}{\|u_n\|}$.

Then $\|v_n\| = 1$ and $F(v_n) \longrightarrow \lambda$ (because $F(\alpha u) = F(u)$).

Since H is reflexive, for a subsequence, still denoted (v_n), one has $v_n \rightharpoonup v$ weakly in H. Since T is compact $Tv_n \longrightarrow Tv$. Then $F(v_n) = \langle Tv_n, v_n\rangle \longrightarrow \langle Tv, v\rangle$ namely $\langle Tv, v\rangle = \lambda$. In particular $v \neq 0$ because $\lambda > 0$

Moreover $\|v\| \leq \liminf \|v_n\| = 1$. Then

$$\lambda = \sup_{u \neq 0} \frac{\langle Tu, u\rangle}{\|u\|^2} \geq \frac{\langle Tv, v\rangle}{\|v\|^2} = \frac{\lambda}{\|v\|^2} \Longrightarrow \|v\| \geq 1$$

Therefore $\|v\| = 1$ and hence $v_n \longrightarrow v$ s- H and $F(v) = \max_{u \in H\setminus\{0\}} F(u)$. Observe that F is Fréchet-differentiable at every $u \in H \setminus \{0\}$, with

$$F'(u)[h] = \frac{2\langle Tu, h\rangle \|u\|^2 - 2\langle u, h\rangle \langle Tu, u\rangle}{\|u\|^4} \qquad \forall h \in H$$

Since v is a maximum point one has $F'(v) = 0$, that is

$$\langle Tv, h\rangle \|v\|^2 - \langle v, h\rangle \langle Tv, v\rangle = 0 \qquad \forall h \in H$$

namely $\langle Tv - \lambda v, h\rangle = 0$ $\forall h$ (because $\|v\| = 1$ and $\langle Tv, v\rangle = \lambda$. Then $Tv = \lambda v$. □

Proof of the theorem

If $T \equiv 0$ then no nonzero eigenvalue, and $\ker T = H$

Hence $H_- = H_+ = \emptyset$ and the conclusion is trivial.

Assume $\exists\, u \in H$ s.t. $\langle Tu, u\rangle > 0$.
Define $\lambda_1^+ = \sup\limits_{\substack{u\in H\\ u\neq 0}} \dfrac{\langle Tu, u\rangle}{\|u\|^2}$ and apply the lemma.

Hence λ_1^+ is an eigenvalue. Let u_1^+ be a corresponding eigenvector with $\|u_1^+\| = 1$. Define $H_1 = (\operatorname{span}\{u_1^+\})^\perp$.

Observe that $T(H_1) \subset H_1$ - in fact, if $u \in H_1$ then $\langle Tu, u_1^+\rangle = \langle u, Tu_1^+\rangle = \lambda_1^+ \langle u, u_1^+\rangle = 0$.
Set $T_1 = T|_{H_1}$. Then T_1 is a compact, self-adjoint operator in the Hilbert space H_1. If $\langle T_1 u, u\rangle \leq 0$ $\forall u \in H_1$ then the search of positive eigenvalues stops here. In this case, set $H_+ = H_1$, observe that $T(H_+^\perp) \subset H_+^\perp$ and study $-T|_{H_+^\perp}$. Otherwise $\langle T_1 u, u\rangle > 0$ for some $u \in H_1$. Then repeat the same argument as before.

In particular $\lambda_2^+ = \sup\limits_{\substack{u\in H_1\\ u\neq 0}} \dfrac{\langle T_1 u, u\rangle}{\|u\|^2}$ is an eigenvalue for T_1, namely for T, and since $H_1 \subset H$, one has $\lambda_2^+ \leq \lambda_1^+$. Let u_2^+ be a corresponding eigenvector with $\|u_2^+\| = 1$. Since $u_2^+ \in H_1$, $\langle u_2^+, u_1^+\rangle = 0$. Define $H_2 = (\operatorname{span}\{u_1^+, u_2^+\})^\perp$ and iterate the argument. This procedure can stop after finitely many steps. In this case $H_+ = \operatorname{span}\{u_1^+, \dots, u_n^+\}$. Otherwise one finds a decreasing sequence $\{\lambda_n^+\}$ of positive eigenvalues with a corresponding sequence of orthonormal eigenvectors $\{u_n^+\}$. We claim that $\lambda_n^+ \to 0$. Indeed $\exists \lim \lambda_n^+ = \lambda$ with $\lambda \geq 0$. The sequence (u_n^+) admits a subsequence still denoted (u_n^+), w-converging to some v. Then $\langle Tu_n^+, u_n^+\rangle \to \langle Tv, v\rangle$ and also $\langle Tu_n^+, u_n^+\rangle = \lambda_n^+ \to \lambda$. Hence $\langle Tv, v\rangle = \lambda$. Fixing $k \in \mathbb{N}$ one has that $\langle u_n^+, u_k^+\rangle = 0$ $\forall n > k$. Then passing to the limit $n \to \infty$ one gets $\langle v, u_k^+\rangle = 0$. This holds $\forall k$. Then $v \in H_n$ and if $\lambda \neq 0$ then $v \neq 0$. Hence

$$\lambda_n^+ = \sup_{\substack{u\in H_n\\ u\neq 0}} \frac{\langle Tu, u\rangle}{\|u\|^2} \geq \frac{\langle Tv, v\rangle}{\|v\|^2} = \frac{\lambda}{\|v\|^2}$$

where $H_n = (\operatorname{span}\{u_1^+, \dots, u_n^+\})^\perp$. Then $\lambda \geq \dfrac{\lambda}{\|v\|^2}$ and thus $\|v\| = 1$ and consequently $u_n^+ \to v$ s-H.

This is impossible, because $\|u_n^+ - u_k^+\|^2 = 2$ if $n \neq k$. Therefore $\lambda = 0$. Setting $H_+ = \overline{\text{span}\{u_n^+\}_n}$ one has that $T(H_+^\perp) \subset H_+^\perp$ and $\langle Tu, u\rangle \leq 0 \ \forall u \in H_+^\perp$. Now one works with $-T$ restricted to $H_+^\perp$ and one obtains negative eigenvalues $\{\lambda_n^-\}$ with a corresponding family of orthonormal eigenvectors $\{v_n^-\}$.

Setting $H_- = \overline{\text{span}\{u_n^-\}_n}$ and $H_0 = (H_+ \oplus H_-)^\perp$ one has that $\langle Tu, u\rangle = 0 \ \forall u \in H_0$. It holds that $T(H_0) \subset H_0$. In fact if $u \in H_0$ then $\forall v \in H_+ \oplus H_-$ one has

$$\langle Tu, v\rangle = \langle Tu, \sum_{n\in I_+} c_n^+ u_n^+ + \sum_{n\in I_-} c_n^- u_n^- \rangle =$$

$$= \sum_{n\in I_+} c_n^+ \langle u, Tu_n^+\rangle + \sum_{n\in I_-} c_n^- \langle u, Tu_n^-\rangle$$

$$= \langle u, \sum c_n^+ \lambda_n^+ u_n^+ + \sum c_n^- \lambda_n^- u_n^-\rangle = 0$$

namely $Tu \in (H_+ \oplus H_-)^\perp$. Moreover $\langle Tu, u\rangle = 0 \ \forall u \in H_0$. Hence the functional $F(u) = \langle Tu, u\rangle$ is constant in H_0. Then $F'(u) = 0 \ \forall u \in H_0$, namely $Tu = 0 \ \forall u \in H_0$, that is $H_0 \subset \ker T$. On the other hand, if $Tu = 0$ then

$$\langle u, u_n^\pm\rangle = \frac{1}{\lambda_n^\pm}\langle u, Tu_n^\pm\rangle = \frac{1}{\lambda_n^\pm}\langle Tu, u_n^\pm\rangle = 0$$

hence $u \in (H_+ \oplus H_-)^\perp$. Therefore $H_0 = \ker T$.

Finally we show that there is no nonzero eigenvalue different from λ_n^+ or λ_n^- - in fact let $\mu \geq 0$ be an eigenvalue and let v be a corresponding eigenvector. Then $\forall n$

$$\langle v, u_n^+\rangle = \frac{1}{\lambda_n^+}\langle v, Tu_n^+\rangle = \frac{1}{\lambda_n^+}\langle Tv, u_n^+\rangle = \frac{\mu}{\lambda_n^+}\langle v, u_n^+\rangle \quad \text{and } \mu \neq \lambda_n^+.$$

Therefore $\langle v, u_n^+\rangle = 0 \ \forall n$, that is $v \in H_n^+ = (\text{span}\{u_1^+, \dots, u_n^+\})^\perp$

and hence $\lambda_n^+ = \sup_{\substack{u\in H_n^+ \\ u\neq 0}} \frac{\langle Tu, u\rangle}{\|u\|^2} \geq \frac{\langle Tv, v\rangle}{\|v\|^2} = \mu$.

If the λ_n^+ are infinite then $\lambda_n^+ \to 0$ and then $\mu = 0$.
If there are finitely many λ_n^+ then $\langle Tu, u\rangle \leq 0 \ \forall u \in H^+$ and in particular $\langle Tv, v\rangle \leq 0 \Rightarrow \mu = 0$. □

Remark Each non zero eigenvalue has finite multiplicity.

Corollary H real or complex Hilbert space with $\dim H = \infty$, $T : H \to H$ linear, compact, self-adjoint operator such that $\langle Tu, u \rangle > 0 \ \forall u \neq 0$. Then $\exists\ (\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}$, $(u_n)_{n \in \mathbb{N}} \subset H$ s.t.

$0 < \lambda_{n+1} \leq \lambda_n \ \forall n \in \mathbb{N}$, $\lambda_n \to 0$ as $n \to \infty$

(u_n) is a Hilbertian basis of H

$T u_n = \lambda_n u_n \quad \forall n$

Proof If λ is an eigenvalue of T with eigenvector u then $\lambda = \frac{\langle Tu, u \rangle}{\|u\|^2} > 0$. Hence by the Hilbert-Schmidt theorem, $H = \overline{\text{span}\{u_n\}}$ with $\langle u_n, u_m \rangle = \delta_{nm}$ and each u_n is an eigenvector corresponding to a positive eigenvalue λ_n. Moreover $\lambda_1 \geq \lambda_2 \geq \dots$ The λ_n's are infinite because $\dim H = \infty$. Hence, according to the Hilbert-Schmidt theorem, $\lambda_n \to 0$. □