Semilinear elliptic problems with critical growth

Let us consider the problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ in dimension $N \geq 3$ and $p > 1$. In view of what proved, we have that:

- for $1 < p < \frac{2N}{N-2}$ there exists a solution, classical if Ω is regular enough.
- for $p \geq \frac{2N}{N-2}$, if Ω is star-shaped, then there is no solution.
- The geometric condition on Ω in the previous non-existence result is essential. In fact, in 1984, using variational methods, J.M. Coron proved existence of solutions for annular-shaped regions and $p = \frac{2N}{N-2}$. Also some existence results in the supercritical case (but close to the critical exponent) are available.

The presence of a threshold on the exponent p distinguishing existence from non-existence for problem (1) is an interesting feature which can be motivated by observing that there is a "competition" between the positive operators $u \mapsto -\Delta u$ and $u \mapsto |u|^{p-1}u$ in the space $H_0^1(\Omega)$. The operator $-\Delta$ is positive in the sense that $\langle u, -\Delta u\rangle_{L^2} = \|\nabla u\|_{L^2}^2 > 0 \ \forall u \in H_0^1(\Omega) \setminus \{0\}$, similarly for the operator $u \mapsto |u|^{p-1}u$, since $\langle u, |u|^{p-1}u\rangle_{L^2} = \|u\|_{L^{p+1}}^{p+1} > 0 \ \forall u \in H_0^1(\Omega) \setminus \{0\}$. For existence of a positive solution, a balance is needed and is reached when p stays below the special value $\frac{N+2}{N-2}$.

Remark 0.1 *Let us consider the problem*

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{N+2}{N-2}} & \textit{in } \Omega \\ u > 0 & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ in dimension $N \geq 3$. If the domain is star-shaped a necessary condition for existence of solutions is that $\lambda \in (0, \lambda_1)$. In 1983 in fact H. Brezis and L. Nierenberg proved that:

- *in dimension $N \geq 4$, for any bounded regular domain Ω in $\mathbb{R}^N$ problem (2) admits a solution for every $\lambda \in (0, \lambda_1)$;*
- *in dimension $N = 3$, for any bounded regular domain $\Omega \subset \mathbb{R}^3$ there is $\lambda^* \in (0, \lambda_1)$ such that problem (2) admits a solution for $\lambda \in (\lambda^*, \lambda_1)$;*
- *considering the three-dimensional ball $B = \{x \in \mathbb{R}^3 \mid |x| < 1\}$, problem (2) admits a solution for $\lambda \in \left(\frac{\lambda_1}{4}, \lambda_1\right)$ and no solution exists if $\lambda \in \left(0, \frac{\lambda_1}{4}\right]$.*

In particular, the solution is found as a multiple of a minimizer for problem $\inf_{u\in M} F_\lambda(u)$*, where*

$$F_\lambda(u) = \int_\Omega (|\nabla u|^2 - \lambda u^2)\,dx \quad \textit{and} \quad M = \{u \in H_0^1(\Omega) \mid \|u\|_{L^{\frac{2N}{N-2}}} = 1\}\,. \tag{3}$$

Hereinafeter we aim to investigate minimization problems defined by

$$S(\Omega) = \inf_{u\in M} \|\nabla u\|_{L^2}^2 \tag{4}$$

where M is defined in (3).

Theorem 0.2 *If* Ω *is a bounded domain in* $\mathbb{R}^N$ *then* $S(\Omega)$ *is not attained and is independent of* Ω*.*

Proof. Let $B_r(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < r\}$ and $B = B_1(0)$. If $u \in H_0^1(B_r(x_0))$ then the function $v(x) = u(rx + x_0)$ belongs to $H_0^1(B)$, and viceversa, with

$$\int_B |\nabla v|^2\,dx = r^{2-N}\int_{B_r(x_0)} |\nabla u|^2\,dx \quad \text{and} \quad \int_B |v|^{2^*}\,dx = r^{-N}\int_{B_r(x_0)} |u|^{2^*}\,dx\,.$$

Hence

$$\frac{\int_B |\nabla v|^2\,dx}{\left(\int_B |v|^{2^*}\,dx\right)^{2/2^*}} = \frac{\int_{B_r(x_0)} |\nabla u|^2\,dx}{\left(\int_{B_r(x_0)} |u|^{2^*}\,dx\right)^{2/2^*}}\,.$$

As a consequence, $S(B_r(x_0)) = S(B)$. Finally, if Ω is any bounded domain in $\mathbb{R}^N$, there exist $x_0 \in \mathbb{R}^N$, $0 < r_1 < r_2$ such that $B_{r_1}(x_0) \subset \Omega \subset B_{r_2}(\Omega)$. Since each function $u \in H_0^1(\Omega)$, by null extension, belongs also to $H_0^1(B_{r_1}(x_0))$, we have that $S(B_{r_2}(x_0)) \le S(\Omega) \le S(B_{r_1}(x_0))$. Hence by the first part of the proof $S(\Omega)$ does not depend on Ω. Moreover it is not attained. Otherwise a minimizer for $S(\Omega)$ in $H_0^1(\Omega)$ would be a minimizer also for $S(B_r(0))$ in $H_0^1(B_r(0))$ for a ball $B_r(0) \supset \Omega$. Therefore we would find a solution for problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} & \text{in } B_r(0) \\ u \ge 0 & \text{in } B_r(0) \\ u = 0 & \text{on } \partial B_r(0) \end{cases}$$

Since $u \not\equiv 0$, by the maximum principle, $u(x) > 0$ for every $x \in B_r(0)$. But this is false, because $u(x) = 0$ for $x \in B_r(0) \setminus \Omega$. □

The situation is different when the domain is the whole space $\mathbb{R}^N$. To this purpose, let us recall the Sobolev inequality:

Theorem 0.3 *In dimension* $N \ge 3$*, there exists* $C_N > 0$ *such that for every* $u \in C_c^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\,dx \le C_N \left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^{\frac{N-2}{N}}\,. \tag{5}$$

Let $2^* = \frac{2N}{N-2}$ and

$$D^1(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \mid \exists \partial_{x_i} u \in L^2(\mathbb{R}^N)\ \forall i = 1, ..., N\}$$

and observe that by density (5) holds also in $D^1(\mathbb{R}^N)$ and then $\|\nabla u\|_{L^2}$ defines a norm in $D^1(\mathbb{R}^N)$. Notice that, by the Sobolev inequality, $D^1(\mathbb{R}^N)$ is contained in $L^{2^*}(\mathbb{R}^N)$, with continuous inclusion.

Remark 0.4 *Differently from the space $H^1(\mathbb{R}^N)$, which is embedded into $L^p(\mathbb{R}^N)$ for all $p \in [2, 2^*]$, the space $D^1(\mathbb{R}^N)$ is embedded into $L^p(\mathbb{R}^N)$ only for $p = 2^*$. This can be viewed by obsarving that*

$$\inf_{\substack{u \in C_c^\infty(\mathbb{R}^N) \\ u \neq 0}} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^p}^2} = 0 \quad \text{if} \quad p \neq 2^*.$$

In fact, fix $C_c^\infty(\mathbb{R}^N) \setminus \{0\}$ and set $u_r(x) = u\big(\frac{x}{r}\big)$. Then

$$\frac{\|\nabla u_r\|_{L^2}^2}{\|u_r\|_{L^p}^2} = r^{(N-2)\left(\frac{1}{2^*} - \frac{1}{p}\right)} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^p}^2} \to 0 \quad \begin{cases} \text{as } r \to \infty & \text{if } p < 2^* \\ \text{as } r \to 0 & \text{if } p > 2^*. \end{cases}$$

Hence, in particular, $D^1(\mathbb{R}^N)$ is larger that $H^1(\mathbb{R}^N)$. Moreover, setting

$$S = \inf_{\substack{u \in D^1(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\, dx\right)^{2/2^*}}$$

one has that $S = S(\Omega)$ for every bounded domain Ω in $\mathbb{R}^N$.

Theorem 0.5 (T. Aubin, 1976; G. Talenti, 1976) *The infimum S is attained in $D^1(\mathbb{R}^N)$ by the function*

$$U(x) = \frac{1}{(1 + |x|^2)^{\frac{N-2}{2}}}.$$

Remark 0.6 *In view of the invariance under dilations and translations of the quotient $\|\nabla u\|_{L^2}^2 / \|u\|_{L^{2^*}}^2$, for every $r > 0$ and $x_0 \in \mathbb{R}^N$ also the function $U_{r,x_0}(x) = U(rx + x_0)$ is a minimizer for S and $S^{\frac{N-2}{4}} U_{r,x_0}$ solves*

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N \\ u > 0 \text{ in } \mathbb{R}^N,\ u \in D^1(\mathbb{R}^N)\,. \end{cases}$$

In fact a solution to the above problem is of the form $u = S^{\frac{N-2}{4}} U_{r,x_0}$ for some $r > 0$ and $x_0 \in \mathbb{R}^N$. Note that the function $v(x) = \big(\frac{N-2}{2}\big)|x|^{-\frac{N-2}{2}}$ is a positive solution to $-\Delta u = u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N \setminus \{0\}$ but does not belong to $D^1(\mathbb{R}^N)$.

Proof (Lieb, 1983). Let $D^1_r = \{u \in D^1(\mathbb{R}^N) \mid u \text{ radial and radially decreasing}\}$. Hence $D^1_r = \{u \in D^1(\mathbb{R}^N) \mid u = u^* \text{ a.e.}\}$, where u^* denotes that spherical decreasing rearrangement of u. Recalling that $\|u\|_{L^p} = \|u^*\|_{L^p}$ if $u \in L^p$ ($p \in [1, \infty]$) and $\nabla u^*\|_{L^2} \leq \|\nabla u\|_{L^2}$, by density, one has that

$$\begin{aligned} S &= \inf \left\{ \frac{\|\nabla u\|^2_{L^2}}{\|u\|^2_{L^{2^*}}} \mid u \in D^1(\mathbb{R}^N) \cap L^2 \cap L^\infty \right\} \\ &= \inf \left\{ \frac{\|\nabla u\|^2_{L^2}}{\|u\|^2_{L^{2^*}}} \mid u \in D^1_r \cap L^2 \cap L^\infty \right\} . \end{aligned}$$

Now make the following change of variable: for $u \in D^1_r$, let $w(t) = e^t v(e^{at})$ where $v(|x|) = u(x)$, and $a = \frac{2}{N-2}$. We have that

$$w'(t) = e^t v(e^{at}) + a e^{(a+1)t} v'(e^{at}) = w(t) + e^{(a+1)t} v'(e^{at})$$

and then

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \sigma_N \int_0^\infty v'(r)^2 r^{N-1} \, dr = \frac{\sigma_N}{a} \int_{-\infty}^\infty (w'(t) - w(t))^2 \, dt$$

where σ_N is the $(N-1)$-dimensional area of the sphere $\{x \in \mathbb{R}^N \mid |x| = 1\}$. Moreover

$$\int_{\mathbb{R}^N} |u|^{2^*} \, dx = \sigma_N \int_0^\infty v(r)^{2^*} r^{N-1} \, dr = a\sigma_N \int_{-\infty}^\infty w(t)^{2^*} \, dt \, .$$

If, in addition, $u \in L^2 \cap L^\infty$, then, on one hand

$$\int_{\mathbb{R}^N} u^2 \, dx \geq \sigma_N \int_0^r v(\rho)^2 \rho^{N-1} \, d\rho \geq \frac{\sigma_N}{N} r^N v(r)^2 \quad \forall r > 0$$

that implies

$$w(t) \leq C e^{-\left(\frac{aN}{2} - 1\right)t} = C e^{-at} \quad \forall t \in \mathbb{R} \, ,$$

and thus $w \in L^p(0, \infty)$ for all $p \geq 1$. On the other hand, $w(t) \leq \|u\|_\infty e^t$ and then $w \in L^p(-\infty, 0)$ for all $p \geq 1$. Hence $w \in L^2(\mathbb{R})$ and since $w' - w \in L^2(\mathbb{R})$, we infer that $w \in H^1(\mathbb{R})$. In particular

$$\int_{-\infty}^\infty w(t) w'(t) \, dt = \frac{w(\infty)^2}{2} - \frac{w(-\infty)^2}{2} = 0$$

because mappings in $H^1(\mathbb{R})$ vanish at infinity. Therefore

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{\sigma_N}{a} \int_{-\infty}^\infty (w'(t)^2 + w(t)^2) \, dt \, .$$

Hence we proved that

$$C_N M_{2^*} \leq S$$

where

$$C_N = \frac{\sigma_N^{1-\frac{2}{2^*}}}{a^{1+\frac{2}{2^*}}} \quad \text{and} \quad M_p = \inf\left\{ \frac{\|u\|_{H^1}^2}{\|u\|_{L^p}^2} \mid u \in H^1(\mathbb{R}),\ u \neq 0 \right\}.$$

Furthermore, if $w \in H^1(\mathbb{R})$ and

$$u(x) = |x|^{-\frac{1}{a}} w(a^{-1} \log |x|)\,, \tag{6}$$

then $u \in D^1(\mathbb{R}^N)$, with

$$\int_{\mathbb{R}^N} |\nabla u|^2\, dx = \frac{\sigma_N}{a} \int_{-\infty}^{\infty} ((w')^2 + w^2)\, dt \quad \text{and} \quad \int_{\mathbb{R}^N} |u|^{2^*}\, dx = a\sigma_N \int_{-\infty}^{\infty} |w|^{2^*}\, dt$$

which implies that $S \leq C_N M_{2^*}$ and if w is a minimizer for M_{2^*} then u defined by (6) yields a minimizer for S. To complete the proof we use:

Lemma 0.7 *For $p > 2$ the infimum M_p is attained in $H^1(\mathbb{R})$ by*

$$\omega(t) = \frac{1}{(\cosh a_p t)^{\frac{1}{a_p}}} \quad \textit{where} \quad a_p = \frac{p-2}{2}.$$

Noting that $a_{2^*} = a$, by Lemma 0.7, S is attained in $D^1(\mathbb{R}^N)$ by $U(x) = |x|^{-\frac{1}{a}} \omega(a^{-1} \log |x|) = (1 + |x|^2)^{-\frac{N-2}{2}}$. □

Proof of Lemma 0.7. The minimization problem defined by M_p admits a minimum point $w \in H^1(\mathbb{R})$ which is an even, positive function, decreasing on $[0, \infty)$, which solves

$$-w'' + w = M_p w^{p-1} \quad \text{on } \mathbb{R}\,. \tag{7}$$

The total energy

$$h(t) = \frac{w'(t)^2}{2} - \frac{w(t)^2}{2} + \frac{M_p w(t)^p}{p}$$

is constant, call it h_0. Since $w(t) \to 0$ as $|t| \to \infty$ (because $w \in H^1(\mathbb{R})$), $w'(t)^2 \to 2h_0$ as $|t| \to \infty$. As $w' \in L^2(\mathbb{R})$ it must be $h_0 = 0$. Since $w'(0) = 0$ (because w is even), one has

$$0 = -\frac{w(0)^2}{2} + \frac{M_p w(0)^p}{p}\,.$$

Then w solves the Cauchy problem given by the differential equation (7) with initial condition $w(0) = \left(\frac{p}{2M_p}\right)^{\frac{1}{p-2}}$, $w'(0) = 0$. By uniqueness we deduce that $w(t) = w(0)(\cosh a_p t)^{-\frac{1}{a_p}}$ where $a_p = \frac{p-2}{2}$. □