Lesson I

CALCULUS OF VARIATION:

AN OVERVIEW

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The basic problem

Let Ω be an open connected subset of $\mathbf{R}^2$. Given:

- a function $F(x,p) : \mathbf{R}^2 \to \mathbf{R}$ of class C^2 on Ω
- four real numbers t_0, t_1, x_0, x_1 with $t_0 < t_1$

find a function $x = \varphi(t) : [t_0, t_1] \to \mathbf{R}$ of class C^1 such that

$$(*) \qquad \varphi(t_0) = x_0, \quad \varphi(t_1) = x_1,$$

$$(**) \qquad (\varphi(t), \varphi'(t)) \in \Omega \text{ for all } t \in [t_0, t_1] \text{ and}$$

$$(***) \qquad \int_{t_0}^{t_1} F(\varphi(t), \varphi'(t))\, dt \; = \min \int_{t_0}^{t_1} F(g(t), g'(t))\, dt$$

where the minimum is taken over all the functions $x = g(t) : [t_0, t_1] \to \mathbf{R}$ of class C^1 satisfying $(**)$, and the same endpoints condition as in $(*)$

Remarks

$\diamondsuit$ Let us denote

$$J(g(\cdot)) = \int_{t_0}^{t_1} F(g(t), g'(t))\, dt$$

$J(\cdot)$ can be thought of as a map $C^1([t_0, t_1], \mathbf{R}) \to \mathbf{R}$. Maps of this type (from a space of functions to $\mathbf{R}$) are often called *functionals*

$\diamondsuit$ A function $\varphi(t)$ where the minimum of $J(\cdot)$ is attained is called an *extremant*

$\diamondsuit$ The problem is formulated in terms of "absolute minimum"; it can be equivalently formulated in terms of "absolute maximum". The search of absolute minima or maxima of a functional is generally referred to as an *optimization* problem

$\diamondsuit$ The problem makes sense also for "local minima", but the formulation requires some more subtle technical details

$\diamondsuit$ The problem can be easily extended to the case $F(t, x, p)$

The problem above implies two sub-problems:

- find conditions for the existence of the minimum (and possibly, uniqueness)
- find some useful characterization of the extremants

In these lessons, we will focus on the second sub-problem; as far as the first sub-problem is concerned, we limit ourselves to report the following theorem.

Theorem. Let $F(x, p)$ be everywhere defined and strictly convex w.r.t. both variables. Then there exists a unique extremant $\varphi(t)$ minimizing the functional $J(\cdot)$.

The form of the functional $J(\cdot)$ to be minimized seems to be very peculiar. However, it covers a large number of interesting problems.

∘ Minimal distance (geodesics). The solution is obvious in a plane and absence of obstacles, but not on a general surface

∘ Brachistochrone (J. Bernoulli 1696, curve of quickest descent) The solution is a cycloid

∘ Surface of revolution with minimal area. The solution is a catenary

∘ Best shape of a rocket (I. Newton 1687, curve of minimum resistance).

∘ Analytic methods in Mechanics: the principle of minimal action

A useful characterization of extremant is provided by the following theorem.

Theorem. Let $\varphi(t)$ be an extremant for $J(\cdot)$, and assume that it is of class C^2. Then, $\varphi(t)$ is a solution of the differential equation

$$F_x(x,x') - \frac{d}{dt}\left[F_p(x,x')\right] = 0 \ . \tag{1}$$

Equation (1) is called the *Euler equation*

F_x, F_p is a simplified notation for partial derivative

Remarks

$\diamondsuit$ Equation (1) plays the same role as Fermat Theorem in the theory of functions with a finite number of variables. Its solutions are called *extremals* of the functional $J(\cdot)$ or *stationary functions*. Theorem above states that any extremant is an extremal function. However, the converse is false, in general. In other words, equation (1) is a necessary, but not sufficient condition for our optimization problem.
$\diamondsuit$ Euler equation is an ordinary differential equation of second order. It is "difficult", since it is in general nonlinear and not in normal form.
$\diamondsuit$ In spite of these intrinsic difficulties, we may expect that it has a general integral containing two arbitrary constants, which can be determined in principle, exploiting the endpoints conditions $(*)$
$\diamondsuit$ An additional difficulty is that $(1) + (*)$ constitute a boundary values problem (not an initial values problem)

$\diamondsuit$ By virtue of time-invariance (i.e., F independent of t) the Euler Equation admits, in general, a first integral

$$F(x, x') = F_p(x, x')x' + A$$

where A is a constant.

This reduced order form of the Euler equation is very useful in applications

The Hamiltonian formalism

The Hamiltonian formalism allows us to transform the Euler equation (under restrictive assumptions) into a system of two equations of first order in normal form.

Let the function $p \mapsto q = F_p(x, p)$ be globally invertible w.r.t. p for each fixed x namely, assume that:

(H) there exists a map $q \mapsto p = \Psi(x, q)$ of class C^1 such that for each x, q

$$q = F_p(x, \Psi(x, q))$$

Define the *Hamiltonian function*

$$H(x, q) = q\Psi(x, q) - F(x, \Psi(x, q))$$

Theorem. Under the assumption (H), a function $\varphi(t)$ is a solution of the Euler equation (1) if and only if the pair $(\varphi(t), \psi(t))$, where

$$\psi(t) = F_p(\varphi(t), \varphi'(t)) \ ,$$

is a solution of the system

$$\begin{cases} x' = H_q(x, q) \\ q' = -H_x(x, q) \end{cases} \tag{2}$$

The variables x, q are called *canonic*, q is also called the *adjoint* variable.
The transformation $(x, p) \leftrightarrow (x, q)$ is called *Legendre transform*

Note that if $(\varphi(t), \psi(t))$ is a solution of (2),

$$H(\varphi(t), \psi(t)) = \text{constant}$$

Extensions of the basic problem

• Several variables. If $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, we will have one Euler equation for each variable x_i. Instead of (1), we will have therefore a system of n ordinary differential equations. The Hamiltonian function will depend on $2n$ variables, and (2) will consist of $2n$ equations.

• Free endpoints. If one (or more) among the four numbers t_0, t_1, x_0, x_1 is not preassigned, it should be considered as a unknown. Say for instance that the free variable is t_1, and update our problem: find the minimum of

$$J(g(\cdot), t_1) = \int_{t_0}^{t_1} F(g(t), g'(t))\, dt$$

where the minimum is taken over all the functions $x = g(t) : [t_0, +\infty) \to \mathbf{R}$ of class C^1 satisfying the endpoints condition in $(*)$ for some $t_1 > t_0$.

The Euler equation remains valid as a necessary condition for optimization, but now we have only three constants which are not enough to determine the general integral.

However, a fourth condition can be determined. In our case:

$$F(\varphi(t_1), \varphi'(t_1)) - \varphi'(t_1) F_p(\varphi(t_1), \varphi'(t_1)) = 0$$

which must be satisfied by any extremant $\varphi(t) : [0, t_1] \to \mathbf{R}$ at the final time t_1.

Conditions of this type are called *transversality conditions*

• Problems with constraints. Let $n > 1$. Sometimes, the search for the minimum or maximum of the functional $J(\cdot)$ is restricted to those functions of class C^1 satisfying additional conditions like:

◦ $$\begin{cases} G_1(x) = 0 \\ \dots \\ G_m(x) = 0 \end{cases}$$ (holonomic constraint) or, more generally,

◦ $$\begin{cases} G_1(x, x') = 0 \\ \dots \\ G_m(x, x') = 0 \end{cases}$$ (nonholonomic constraint)

Here, $G_1, \dots, G_m$ $(m < n)$ are given functions of class C^1.

Examples of constrained problems:

- Geodesics on a surface (for instance, a sphere)
- Isoperimetric problems (for instance, Dido problem)
- Rolling wheel (without sliding)

In the presence of constraints the Euler (system of) equations is no more a necessary condition.

Necessary conditions for a problem with m constraints can be obtained (under a suitable independency condition) by introducing m new variables, called *Lagrange multipliers*.

Let $\tilde{F}(x,p,\lambda) = F(x,p) + \sum_{i=1}^{m} \lambda_i G_i(x,p)$.

Theorem. Let $\varphi(t) = (\varphi_1(t),\ldots,\varphi_n(t))$ be a constrained extremant of the given functional $J(\cdot)$. Then, there exists a function $\lambda(t) = (\lambda_1(t),\ldots,\lambda_m(t)) : [t_0,t_1] \to \mathbf{R}^m$ of class C^1 such that the pair $(\varphi(t),\lambda(t))$ is a free extremal of the functional

$$\tilde{J}(g(\cdot),\ell(\cdot)) = \int_{t_0}^{t_1} \tilde{F}(g(t),g'(t),\ell(t))\,dt$$

Note that the system of Euler equations for $\tilde{J}$ combines differential equations of second order, differential equations of first order (case of nonholonomic constraints) or numerical (nondifferential) equations (case of holonomic constraints), the equations of the constraints being incorporated into the system of Euler equations.