Lesson II

CONTROL SYSTEMS

and

THE OPTIMAL CONTROL PROBLEM

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Mathematical and physical methods for space sciences

We consider a mathematical model of the form

$$\text{(CS)} \qquad \dot{x} = f(x, u)$$

where $x \in \mathbf{R}^n$ is the state variable and $u \in \mathbf{R}^m$ is an external input, that we interpret as the control variable. We assume that $f(x, u) : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is of class C^1. Traditionally, we adopt the notation $\dot{x} = \frac{dx}{dt}$.

Let $U \subset \mathbf{R}^m$ be given ($U \neq \emptyset$). A control $u = u(t)$ is admissible if it is defined, piecewise continuous (right continuous), and such that $u(t) \in U$ for $t \in [0, +\infty)$. The set of admissible controls is denoted by $\mathcal{U}$.

The pair ((CS),$\mathcal{U}$) will be referred to as a control system.

For each $u(\cdot) \in \mathcal{U}$ and each $\bar{x} \in \mathbf{R}^n$, there exists a unique (local) solution of (CS), denoted $x(t; \bar{x}, u(\cdot)) = x(t)$ such that $x(0) = \bar{x}$. We implicitly assume appropriate conditions guaranteeing that the solutions of (CS) are actually defined on the whole interval $[0, +\infty)$.

A solution $x(t)$ of (CS) represents the time evolution of the state variable under the action of the control $u(t)$. The image on $\mathbf{R}^n$ of a solution is called a (controlled) trajectory.

A control problem consists in studying existence and characterizations of admissible controls which enable the system to accomplish a prescribed goal.

The controllability problem. Given an "initial" state $\bar{x}$ and a "final" state $\underline{x}$ find $u(\cdot) \in \mathcal{U}$ in such a way that the corresponding solution $x(t)$ satisfies

$$x(0) = \bar{x} , \qquad x(T) = \underline{x} \tag{1}$$

for some $T > 0$.

Definition. Given $\bar{x} \in \mathbf{R}^n$ and $T > 0$, the set of points $\underline{x} \in \mathbf{R}^n$ for which there exists $u(\cdot) \in \mathcal{U}$ such that (1) hold is denoted by $R(\bar{x}, T)$ and it is called the reachable set from $\bar{x}$ at time T.

The optimal control problem. Assume that the following data are given.

(i) A control system $((CS), \mathcal{U})$
(ii) a pair of points $\bar{x}, \underline{x} \in \mathbf{R}^n$
(iii) a function $f_0(x,u) : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ of class C^1.

and let

$$J(T, u(\cdot)) = \int_0^T f_0(x(t), u(t))\, dt$$

where $x(t) = x(t; \bar{x}, u(\cdot))$.

Find $u^*(\cdot) \in \mathcal{U}$ and $T^* > 0$ such that the controllability conditions (1) are met and, moreover

$$J(T^*, u^*(\cdot)) = \int_0^{T^*} f_0(x^*(t), u^*(t))\, dt \;= \min J(T, u(\cdot))$$

where the minimum is taken over all $T > 0$ and all the admissible controls $u(\cdot) \in \mathcal{U}$ such that the corresponding solution of (CS) meets the controllability conditions (1).

The triplet $(T^*, u^*(\cdot), x^*(\cdot))$ (where $x^*(t) = x(t, \bar{x}, u^*(\cdot))$) is called an optimal triplet

T^*, $u^*(\cdot)$, $x^*(\cdot)$ are called, respectively, the optimal time, the optimal control and the optimal trajectory.

Remarks

$\diamondsuit$ The optimal control problem is a generalization of the calculus of variation problem. Indeed, when $n = m$ and $f(x,u) = u$, the dynamics equation (CS) reduces to

$$\dot{x} = u$$

so that the functional to be minimized can be rewritten as

$$J(T, x(\cdot)) = \int_0^T f_0(x(t), \dot{x}(t))\, dt$$

$\diamondsuit$ The **minimum time problem** is a particular case of the optimal control problem. Indeed, taking $f_0(x,u) = 1$ we get $J(T, u(\cdot)) = T$.

$\diamondsuit$ Particular instances of the optimal control problem arise when the final time T is preassigned.

We will consider also problems where $T = +\infty$. In these cases, the second endpoint condition is replaced by

$$\lim_{t \to +\infty} x(t) = \underline{x}$$

If $T = +\infty$, the convergence of the integral which defines J should be added to the requirements.

Optimal control and Calculus of variation. An approach to the optimal control problem based on the methods of the Calculus of Variation is possible, but it requires severe restrictions.

(R1) Assume $U = \mathbf{R}^m$ (no constraints on the values of the control functions)

(R2) Assume that only control functions $u(\cdot) \in C^1$ are admissible

Let us treat u as an independent variable and the dynamic equation (CS) as a nonholonomic constraint. Let us introduce the Lagrange multiplier $\lambda = (\lambda_1, \ldots, \lambda_n)$ and the function

$$L(x, p, u, \lambda) = f_0(x, u) + \lambda \cdot [p - f(x, u)]$$

WARNING: It is convenient, for notational consistency in later use, to treat λ as a row-vector. Recall that also the gradient of a function is considered a row-vector.

The system of Euler equations consists of three blocks of equations

$$\begin{cases} L_x(x,u,\lambda,\dot{x}) - \dfrac{d}{dt} L_{\dot{x}}(x,u,\lambda,\dot{x}) = 0 & (n \text{ eq.s}) \\ \\ L_u(x,u,\lambda,\dot{x}) - \dfrac{d}{dt} L_{\dot{u}}(x,u,\lambda,\dot{x}) = 0 & (m \text{ eq.s}) \\ \\ L_\lambda(x,u,\lambda,\dot{x}) = 0 & (n \text{ eq.s}) \end{cases}$$

that is

$$
\text{(E)} \quad \begin{cases} \dfrac{\partial f_0}{\partial x}(x,u) - \lambda \ \dfrac{\partial f}{\partial x}(x,u) - \dot{\lambda} = 0 \\[2ex] \dfrac{\partial f_0}{\partial u}(x,u) - \lambda \ \dfrac{\partial f}{\partial u}(x,u) = 0 \\[2ex] \dot{x} - f(x,u) = 0 \end{cases}
$$

Note that the second block does not contain differential equations (since $\dot{u}$ does not appear explicitly in L) and that the third block reduce to (CS).

The system above can be written in Hamiltonian form. To chose the appropriate Hamiltonian function, we argue by analogy.

In the classical calculus of variation, for the problem of minimizing

$$J(g(\cdot)) = \int_{t_0}^{t_1} F(g(t), g'(t))\, dt$$

the Hamiltonian function was defined as

$$\text{(H1)} \qquad H(x,q) = q \cdot \Psi(x,q) - F(x, \Psi(x,q))$$

$(q \in \mathbf{R}^n)$ where $p = \Psi(x,q)$ is the inverse of the map

$$\text{(1)} \qquad p \mapsto q = F_p(x,p)$$

Unfortunately, in our case the map $p \mapsto L_p(x, p, u, \lambda)$ is constant with respect to p, and hence not invertible: indeed

$$L_p(x, p, u, \lambda) = \lambda \tag{2}$$

However, "forcing" the notation, we can rewrite (H1) as

$$H(x, q) = q \cdot \dot{x} - F(x, \dot{x}) \tag{H2}$$

Moreover, taking into account (1), (2), it is natural to identify $q = \lambda$.

Finally, recalling the constraint equation and the additional variable u, we are led to define the Hamiltonian function for the problem at hand as

$$\text{(H3)} \quad H(x, u, \lambda) = \lambda \cdot \dot{x} - f_0(x, u) - \lambda \cdot [\dot{x} - f(x, u)] =$$

$$= \lambda \cdot f(x, u) - f_0(x, u)$$

The third and first block of equations in (E) can be now rewritten as

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial \lambda}(x, u, \lambda) \\ \\ \dot{\lambda} = -\dfrac{\partial H}{\partial x}(x, u, \lambda) \end{cases}$$

while the second one becomes

$$\frac{\partial H}{\partial u}(x, u, \lambda) = 0$$

This last equation can be clearly interpreted as a stationarity condition for the map $u \mapsto H(x, u, \lambda)$ for fixed (x, λ)

Under the restriction (R1), (R2), the problem has been therefore reduced to a finite dimensional problem, parameterized by (x, λ).

The following procedure for solving our problem can be devised.

Step 1. Solve the Hamiltonian system with the controllability conditions, taking for the moment u as a parameter in order to find $(x(t,u), \lambda(t,u))$

Step 2. Find $u(t)$ solving the equation

$$\frac{\partial H}{\partial u}(x(t,u), u, \lambda(t,u)) = 0$$

Minimum time. As far as the Minimum Time problem is concerned, a different approach can be proposed. This too, requires some restrictions.

The Dynamic Programming method (R. Bellman, 1950) rests on a very general principle: it basically states that "segments" of optimal trajectories still are optimal trajectories.

Let the system (CS) be given under the usual assumptions. Set (w.l.g.)

$$\underline{x} = 0$$

Assume further that each initial state $\bar{x} \in \mathbf{R}^n$ can be steered to the origin by an admissible control. Define

$$T(\bar{x}) = \inf\{t : \exists u(\cdot) : x(t; \bar{x}, u(\cdot)) = 0\}$$

$T(x)$ is called the value function. If U is compact and $\bar{x} \neq 0$, we have $T(\bar{x}) > 0$.

The required restrictions are:

$(R3)$ the minimum time problem has a solution $\forall \bar{x} \in \mathbf{R}^n$
$(R4)$ $T(\cdot) \in C^1(\mathbf{R}^n \setminus \{0\})$

Lemma. Let $x^*(t) : [0, T(\bar{x})] \to \mathbf{R}^n$ be an optimal trajectory, steering in minimum time the initial state $\bar{x} \neq 0$ to the origin. Let moreover $0 < \tau < T(\bar{x})$ and $y = x^*(\tau)$. Then, the restriction of $x^*(t)$ on the interval $[\tau, T(\bar{x})])$ (suitably translated) is optimal for the problem of steering the point y to the origin in minimum time.

Existence of optimal solutions is guaranteed, for instance, if the system is linear i.e., $f(x, u) = Ax + Bu$, and U is compact, convex, and $0 \in \operatorname{Int} U$.

Using the Lemma, and computing the directional derivative of $T(\cdot)$ along an optimal trajectory issued from an arbitrary initial state x, we get the so-called Bellman equation

$$-\nabla T(x) f(x,u^*) = \max_{u \in U} \Big[-\nabla T(x) f(x,u) \Big] = 1$$

where u^* denotes the value taken at the initial time by the corresponding optimal control.

Note that the Bellman equation represents a necessary condition for optimality.

There is a version of the Dynamic programming method for discrete time systems: in this case, the restrictions about the differentiability of the value function is not necessary.